Volume of a Football

Teacher Notes and Answers

7 8 9 10 11 **12**

TI-Nspire™ Investigation Student 50 min

Introduction

In this activity you will determine the volume of an AFL™ football using calculus.

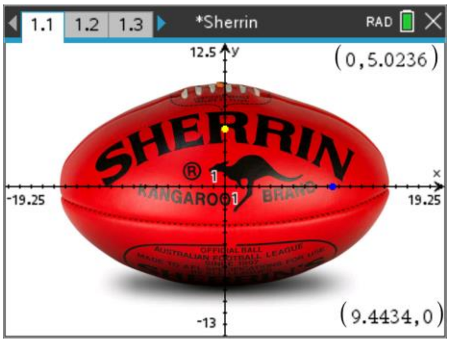
Modelling the ball

Open the TI-Nspire document: Sherrin™.

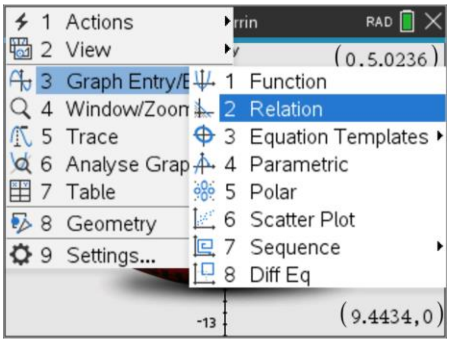
The document contains a picture of an official AFL football. The axis scale has been set to accurately represent the dimensions of the ball where 1 unit on the scale corresponds to 1cm.

Two points have been plotted to help you identify the size of the ball.

Drag the yellow and blue dots to the respective edges of the ball to help determine the size.



Change the graph entry type to: Relation

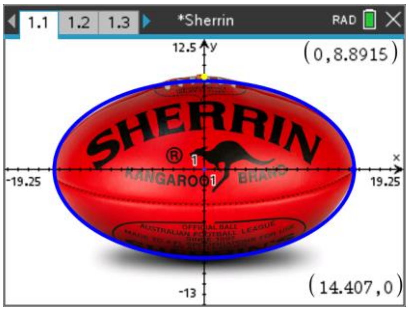


Question: 1

Use your measurements from the points to help determine an appropriate equation for the shape of the ball.

Answer: Answers will vary slightly. The points fit the ball at: $x \approx 14.5$ and $y \approx 8.8$. An appropriate equation to model the ball is an ellipse. Given the measurements (above) and assuming the ball is centred at the origin, a suitable equation would be:

$$\frac{x^2}{14.5^2}+\frac{y^2}{8.8^2}=1$$



Teacher Notes: Students may wish to store the values 14.5 and 8.8 in parameters 'a' and 'b' respectively. This encourages generality and will also help relate other equation forms such as the parametric form for an ellipse.

Author: P. Fox

Question: 2

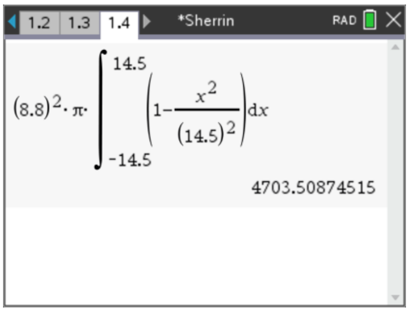
Use calculus to determine the volume of the ball: $V = \pi \int y^2 dx$

Answer: The question states to 'use calculus', given the relatively simplistic integration, students should use 'by-hand' calculations for this result.

$$V = 8.8^2 \pi \int_{-14.5}^{14.5} \left(1 - \frac{x^2}{14.5^2}\right) dx$$

$$V = 8.8^2 \pi \left[x - \frac{x^3}{3 \times 14.5^2} \right]_{-14.5}^{14.5}$$

$$V \approx 4703 cm^3$$



Teacher Notes: Students should be encouraged to include units and recognise that the result is a volume measured in either cm^3 or litres.

Question: 3

Use a practical method to measure the actual volume of a regulation size AFL football and compare the results to those obtained using calculus.

Answer: Methods may vary, most will involve placing the ball in water. Students should check that the ball is the appropriate size by first measuring the circumference. The smaller circumference (y – z axis) should be approximately 55cm and the larger circumference (x – z axis) should be approximately 72.5cm.
A sample used by the author resulted in measurements:

55cm (smaller circumference) & 73cm (larger circumference)

Different inflations will impact these results slightly. Replica balls may not be as accurate.

Placing half the ball in a bucket completely full of water will force water to spill out of the bucket. Use a measuring jug to refill the bucket. Placing half the ball in a bucket has several advantages:

- The valve does not have to be immersed
- It is easier to keep hands out of the water (which would impact the measurement)
- A standard bucket can be used

The sample used by the author resulted in a volume of approximately 2.4 litres being displaced, making the entire volume approximately 4.8 litres, very close to the computed value of 4.7 litres.

Question: 4

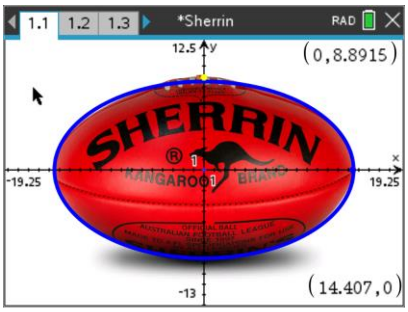
Express your equation for the AFL football in parametric form:

$x(t) = a \cdot \cos(t)$ and $y(t) = b \cdot \sin(t)$

Graph your equations using parametric mode to ensure they match the ball.

Answer: The 'a' and 'b' values from the Cartesian equation can be used here.

Teacher Notes: The screen opposite shows how well the ellipse models the shape of the ball. The parametric and cartesian equations should be visually identical.



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Question: 5

The equation for the solid of revolution in Cartesian form can easily be changed to suit parametric equations:

$$V = \pi\int_{x_1}^{x_2} y^2 dx = \pi\int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$$

a) Show that the volume is equal to: $-ab^2\pi\int_{t_1}^{t_2} \sin(t)^3 dt$

Answer: From question 4: $y^2 = b^2\sin^2(t)$ and $\frac{dx}{dt} = -a\sin(t)$, therefore: $V = -ab^2\pi\int_{t_1}^{t_2} \sin(t)^3 dt$

b) Determine the values of t_1 and t_2.

Answer: $t_1 = 0$ and $t_1 = \pi$

c) Determine the volume and compare the result to the previous calculation using the Cartesian equation.

Answer: Students may use their calculator or by hand: (General solution shown below)

$$\begin{aligned}
V &= -ab^2\pi\int_0^{\pi} \sin^3(t)dt \\
&= -ab^2\pi\int_0^{\pi} \left((1-\cos^2(t))\sin(t)\right)dt \\
&= -ab^2\pi\int_0^{\pi} \left(\sin(t)-\cos^2(t)\sin(t)\right)dt \\
&= -ab^2\pi\int_0^{\pi} \sin(t)dt - ab^2\pi\int_0^{\pi} \cos^2(t)\sin(t)dt \\
&= ab^2\pi\left[\cos(t)\right]_0^{\pi} + ab^2\pi\int_1^{-1} u^2 du \\
&= 2ab^2\pi + ab^2\pi\left[\frac{u^3}{3}\right]_1^{-1} \\
&= \frac{4ab^2\pi}{3}
\end{aligned}$$

Sherrin is the official football of the AFL, they have a sizing chart on their website: https://www.sherrin.com.au/size-chart

The dimensions of a size 5 football (Game Ball) is given by two circumference measurements. The smaller circumference (55cm) is for the circular cross-section and the larger (72.5cm) for the elliptical cross section.

and the j - k plane is an ellipse.

Question: 6

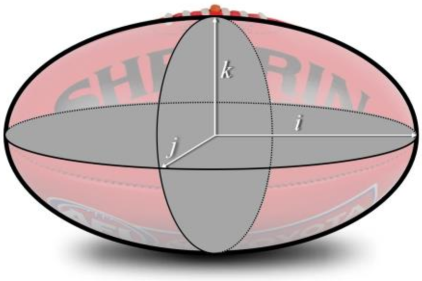
Determine the size of the semi-minor axis given the small circumference is 55cm.

Answer:

The semi-minor axis is the radius of the circle:

$2\pi r = 55$

$r \approx 8.75$



The i - j plane is an ellipse.
The j - k plane is a circle.

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A “formula” does not exist for the circumference of an ellipse; however it is possible to use calculus to determine the arc length.

Cartesian form: $L=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

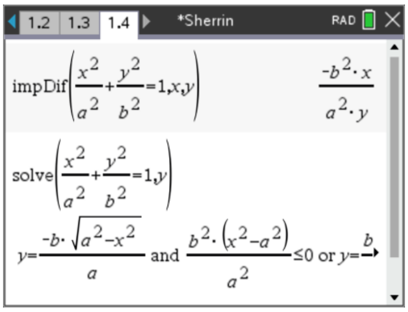
Parametric form: $L=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

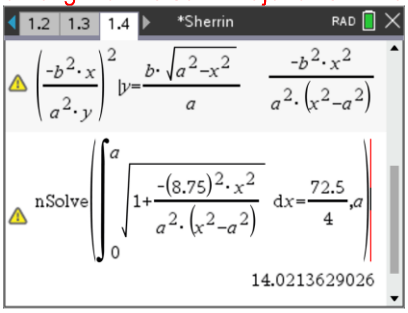
We know that the long circumference (arc length) for a match football is 72.5cm, what we need to determine is a more accurate measurement for the semi-major axis length.

Question: 7

Determine a more precise measurement for the semi-major axis length using the Cartesian model and arc-length formula, combined with the semi-minor axis measurement obtained in question 6.

Answer:
Note: Students can use the approximate result for question 6 rather than exact as the arc-length is an approximated value.
Students use their calculator to ‘solve’ for length of the semi-major axis in their equation.



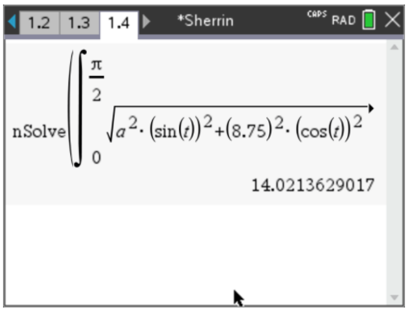


The result is close to the value obtained using the image provided on page 1.1 of the Sherrin TNS file.

Question: 8

Determine a more precise measurement for the semi-major axis length using the parametric model and arc-length formula, combined with the semi-minor axis measurement obtained in question 6.

Answer:
Notice that this approach is much simpler and the calculator generates an answer much quick since the terminals do not include a parameter.



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3D Modelling

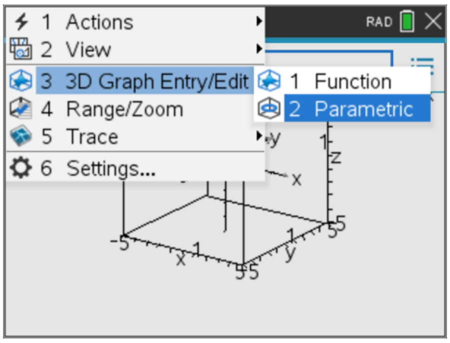
Insert another Graph application into your document.

Change the view to 3D.

menu > **View** > **3D Graphing**

Once you have entered the 3D graphing mode, change to parametric.

menu > **3D Graph Entry/Edit** > **Parametric**

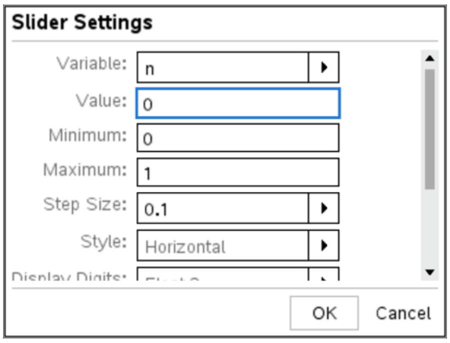


Insert a slider into your 3D graph:

menu > **Actions** > **Insert Slider**

Set the slider variable to: ***n***

Match the settings shown opposite. Also check the box for 'minimised' and uncheck the box for show variable and show scale.

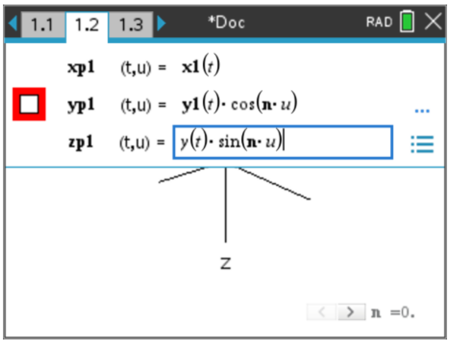


It's time to enter the equation for the graph. For this step your equation must have already been produced in parametric form in the 2D graphing page. (Same problem)

Once these expressions have been entered, the parameters need to be set. Press **TAB** to enter the parameters.

$T_{min} = 0$ $T_{max} = \pi$

$U_{min} = 0$ $U_{max} = 2\pi$



To make the view 'clearer', axis values and the 3D box can be hidden.

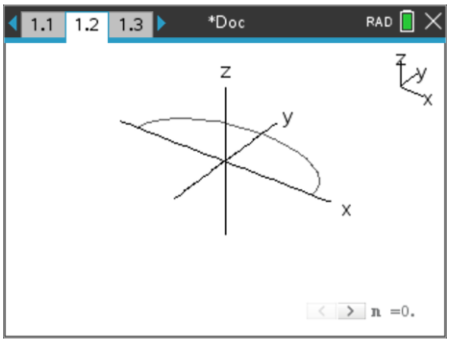
menu > **View** > **Hide Box**

Similarly, with end values:

menu > **View** > **Hide Box End Values**

The arrow keys can be used to rotate the view. Press X, Y or Z to view along a specific axis or A to auto-rotate.

Once you are happy with the display, click on the slider to see the relation rotate around the x axis, then try rotating the view!



Question: 9

Sherrin have a range of football sizes on their website. Select one of the smaller sizes and determine its volume, based on the dimensions provided on the website.

Answer: Answers will vary depending on the selection. A general solution is available in the TI-Nspire file: Teacher General Solution. Change the small and large circumference to see the volume automatically calculated and the graph (3D) automatically generated.

Author: P. Fox