

Activity 18

Exploring Infinite Series

Introduction

One ancient mathematical puzzle is sometimes called the Racecourse Paradox.

A person is running a 1-mile racecourse. Of course, the runner must complete half the course ($1/2$ mile) before completing the entire course. But when the first half of the course is completed, the runner must complete half of what remains ($1/4$ mile) before finishing, and then complete half of what remains after that ($1/8$ mile) before finishing, and so on. Because there are infinitely many distances that the runner must complete before finishing the race, how can the runner ever hope to complete the entire course?

Our everyday experience convinces you that the runner will certainly complete the race, yet adding up infinitely many positive distances would reasonably seem to sum up to an infinite total, or at least would take forever to actually compute!

Exploration

In symbols, the question behind the Racecourse Paradox is whether it makes sense to add the infinitely many terms

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

and obtain a finite sum, namely 1.

Such an infinite summation is called an *infinite series*. Another notation that could be used for the infinite series in the Racecourse Paradox is

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

where the capital Greek letter *sigma* denotes a sum and the infinitely many terms are represented by substituting $n = 1, 2, 3, \dots$ into the expression $\frac{1}{2^n}$.

Objectives

- Identify a geometric series
- Determine convergence and sum of geometric series
- Identify a series that satisfies the alternating series test
- Use a graphing handheld to approximate the sum of a series

Materials

- TI-84 Plus / TI-83 Plus

The mathematical resolution of the Racecourse Paradox does not involve actually adding up infinitely many terms, but rather by considering the infinite series as representing an infinite sequence of *partial sums*, each of which has only a finite number of terms. The sequence of partial sums corresponding to

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is}$$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

$$\vdots$$

The sum of the infinite series is defined as the limit of the corresponding sequence of partial sums. In this case,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^n}\right) = 1$$

and the series *converges*. An infinite series whose sequence of partial sums has no limit is a series that *diverges*.

The infinite series encountered in the Racecourse Paradox is an example of a *geometric series*. A geometric series is one in which the ratio of any term to the one preceding it is always the same. In other words, there is a factor r (the constant ratio) such that each term in the series is r times the term immediately before it.

You can see that the ratio is $r = \frac{1}{2}$ for the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots$$

Two other examples of geometric series are

$$5 - \frac{15}{4} + \frac{45}{16} - \frac{135}{64} + \dots$$

and

$$-2 - \frac{10}{3} - \frac{50}{9} - \frac{250}{27} - \dots$$

The ratio for the first example is $r = \frac{-3}{4}$,

and the ratio for the second example is $r = \frac{5}{3}$.

In general, any geometric series will have the form

$$a + ar + ar^2 + ar^3 + \dots + ar^n + \dots$$

where a is the first term in the series and r is the constant ratio. In summation notation, a geometric series is written as

$$\sum_{n=0}^{\infty} ar^n$$

Note that here the index n starts at $n = 0$, so the first term is $ar^0 = a \cdot 1 = a$.

It is not only easy to determine whether geometric series converge or diverge, but their exact sums can also be computed when they converge.

The partial sums of a geometric series

$$\sum_{n=0}^{\infty} ar^n \text{ are}$$

$$S_0 = a$$

$$S_1 = a + ar$$

$$S_2 = a + ar + ar^2$$

$$S_3 = a + ar + ar^2 + ar^3$$

$$\vdots$$

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^n$$

$$\vdots$$

To determine whether the series converges, decide whether

$$\lim_{n \rightarrow \infty} S_n$$

exists. If the limit exists, then that value is the sum of the series.

S_n can be expressed conveniently, as shown below.

First, multiply S_n by r to get

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n + ar^{n+1}$$

Now, subtract rS_n from S_n :

$$S_n - rS_n = (a + ar + ar^2 + ar^3 + \dots + ar^n) - (ar + ar^2 + ar^3 + \dots + ar^n + ar^{n+1})$$

Rearranging the terms on the right side of the equation,

$$S_n - rS_n = a - ar + ar - ar^2 + ar^2 - ar^3 + ar^3 - \dots - ar^n + ar^n - ar^{n+1} = a - ar^{n+1}$$

If $r \neq 1$, then S_n can be solved for by factoring the left side of the equation and dividing both sides of the equation by $(1 - r)$.

$$S_n = \frac{a - ar^{n+1}}{1 - r} = \frac{a(1 - r^{n+1})}{1 - r}$$

This form is used to find a limit for S_n . If $-1 < r < 1$, then

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{a(1 - r^{n+1})}{1 - r} \right) = \frac{a}{1 - r}$$

because r^{n+1} approaches 0 as n increases. If either $r > 1$ or $r < -1$, then the absolute value of r^{n+1} will grow without bound as n increases, and the limit will not exist.

Two special values of r still need to be addressed. If $r = 1$, then

$$S_n = a + a + a + \dots + a = (n + 1)a$$

will also grow without bound as n increases, and the limit will not exist.

If $r \neq -1$, then

$$S_n = a - a + a - a + \dots + (-1)^n a$$

is equal to 0 if n is odd and is equal to a if n is even, so the partial sums oscillate in value, and the limit still does not exist.

In summary, the geometric series

$$\sum_{n=0}^{\infty} ar^n$$

converges and has sum $\frac{a}{1-r}$ if $-1 < r < 1$, and diverges if $r \geq 1$ or $r \leq -1$.

Returning to the prior two examples

$$5 - \frac{15}{4} + \frac{45}{16} - \frac{135}{64} + \dots$$

converges to the sum

$$\frac{5}{1 - (-3/4)} = \frac{5}{7/4} = 5 \cdot \frac{4}{7} = \frac{20}{7}$$

because the first term is $a = 5$ and the ratio $r = -\frac{3}{4}$ is between -1 and 1 .

As for the second example,

$$-2 - \frac{10}{3} - \frac{50}{9} - \frac{250}{27} - \dots$$

diverges because the ratio $r = \frac{5}{3}$ is greater than 1 .

Keep in mind that an infinite series represents a sequence of partial sums. On the home screen, the **seq** command generates a list based on a formula, an index variable, a starting value, and an ending value. The **sum** command sums the terms in a list. Used together, you can calculate partial sums.

An example follows.

On the home screen, enter **seq(1/2^K,K,1,10)**.

Note: The **seq** command can be found in either the **CATALOG** or in the **LIST Menu** under **OPS**.

```
seq(1/2^K,K,1,10)
)
(.5 .25 .125 .0...
```

To convert the entries in the list to fraction form, select **1:Frac** from the **MATH Menu**.

```
seq(1/2^K,K,1,10)
)
(.5 .25 .125 .0...
Ans>Frac
(1/2 1/4 1/8 1/...
```

Use the **sum** command to find the sum of the ten terms in the list.

```
seq(1/2^K,K,1,10)
)
(.5 .25 .125 .0...
Ans>Frac
(1/2 1/4 1/8 1/...
sum(Ans)
.9990234375
```

This results in the value of

$$\sum_{n=1}^{10} \frac{1}{2^n}$$

the tenth partial sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

The sequence mode on the graphing handheld is a good environment for investigating infinite series with tables and graphs. After you select the **Seq MODE**, input the values shown in the screenshot in the **Y=** editor:

```
Plot1 Plot2 Plot3
nMin=1
:u(n)@sum(seq(1/
2^K,K,1,n))
u(nMin)@1/2
:v(n)=
v(nMin)=
:w(n)=
```

In sequence mode, the independent variable key $\boxed{X,T,\theta,n}$ produces the index variable n . Again, the **sum** command sums up the terms in a list, and the **seq** command generates a list based on a formula, an index variable, a starting value, and an ending value.

Used together in this way, $u(n)$ represents the n th partial sum of our series.

$$\sum_{n=1}^{10} \frac{1}{2^n}$$

View the **TABLE** to see a list of results for these partial sums for different values of n .

| n | $u(n)$ | |
|-----|--------|--|
| 0 | ERROR | |
| 1 | .5 | |
| 2 | .75 | |
| 3 | .875 | |
| 4 | .9375 | |
| 5 | .96875 | |
| 6 | .98438 | |

$n=0$

To see these results displayed graphically (with values of n on the horizontal axis), first check the **FORMAT Menu** to make sure that **Time** is selected. Then set up the viewing window parameters as shown.

```

WINDOW
nMin=1
nMax=10
PlotStart=1
PlotStep=1
Xmin=0
Xmax=5
↓Xscl=1

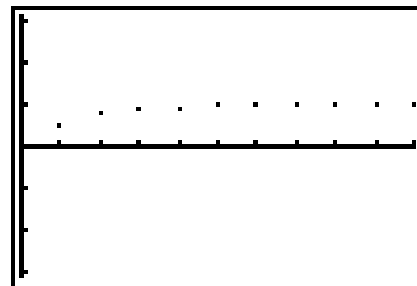
```

```

WINDOW
↑PlotStep=1
Xmin=0
Xmax=10
Xscl=1
Ymin=-3.1
Ymax=3.1
Yscl=1

```

Select **Dot** mode, and press $\boxed{\text{GRAPH}}$ to see the partial sums plotted.



1. Write the geometric series

$$5 - \frac{15}{4} + \frac{45}{16} - \frac{135}{64} + \dots$$

in summation notation.

Use the graphing handheld to generate a list of the first eleven terms ($n = 0$ to $n = 10$) in both decimal and fraction form, and calculate the partial sum for $n = 10$. Then enter the sequence of partial sums in **Y=** editor, and plot each of the partial sums for $n = 0$ to $n = 10$ for the series.

2. Write the geometric series

$$-2 - \frac{10}{3} - \frac{50}{9} - \frac{250}{27} - \dots$$

in summation notation.

Use the graphing handheld to generate a list of the first eleven terms ($n = 0$ to $n = 10$) in both decimal and fraction form, and calculate the partial sum for $n = 10$. Then enter the sequence of partial sums in the **Y=** editor, and plot each of the partial sums for $n = 0$ to $n = 10$ for the series.

3. Explain why the series

$$\frac{3}{2} - \frac{1}{2} + \frac{1}{6} - \frac{1}{18} + \dots$$

is a geometric series. Does it converge? If so, find its sum.

Use the graphing handheld to generate a list of the first eleven terms ($n = 0$ to $n = 10$) in both decimal and fraction form, and calculate the partial sum for $n = 10$. Then enter the sequence of partial sums in the **Y=** editor, and plot each of the partial sums for $n = 0$ to $n = 10$ for the series.

4. Explain why the series

$$\frac{1}{144} + \frac{1}{96} + \frac{1}{64} + \frac{3}{128} + \dots$$

is a geometric series. Does it converge? If so, find its sum.

Use the graphing handheld to generate a list of the first eleven terms ($n = 0$ to $n = 10$) in both decimal and fraction form, and calculate the partial sum for $n = 10$. Then enter the sequence of partial sums in the **Y=** editor, and plot each of the partial sums for $n = 0$ to $n = 10$ for the series in **GRAPH**.

5. Explain why the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

is *not* a geometric series. Do you think it converges?

Use the graphing handheld to find the partial sum for $n = 10$, 20 , and 50 in [2nd] [TABLE]. Then plot each of the partial sums for $n = 0$ to $n = 100$ for the series in **GRAPH**.

The infinite series in Question 5 is an important example called the *harmonic series*. In summation notation, the harmonic series is written as

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

and it *diverges*, although very slowly. The graph of its partial sums should remind you of the graph of a logarithmic function. Logarithmic functions also grow without bound, and also very slowly.

Infinite series like the harmonic series show why relying on computation of partial sums to decide whether an infinite series converges is not practical. In the case of the harmonic series, the individual terms certainly approach 0.

When n becomes large enough, $\frac{1}{n}$ becomes smaller than the precision limits that the graphing handheld can detect.

However, the contributions these small terms make to the total are enough to make the partial sums keep growing without bound.

With a geometric series, there is not only a test telling to identify whether it converges or diverges but also a formula that computes the exact sum of the series (in other words, the limit of the partial sums) when it does converge.

There are many important series that are not geometric, but it is often very difficult to find the exact sum of a convergent series. The next best thing is to have a test that at least identifies whether a given series converges or diverges. Some tests also give a bound on the error that would occur if a partial sum as an approximation to the sum of the whole series were used if the series were convergent.

Series that have terms that alternate in sign (positive and negative) provide us with a simple test that provides both an answer to the convergence question and an error bound.

If

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

is an alternating series such that $a_{n+1} \leq a_n$ for all n (the size of the terms decrease in absolute value) and

$$\lim_{n \rightarrow \infty} a_n = 0$$

then this alternating series converges.

In this situation, the N th partial sum

$$\sum_{n=0}^N (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \dots \pm a_N$$

approximates the sum of

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

with an error no larger than $|a_{N+1}|$, the absolute value of the first term in the remainder of the series.

Note that this test does not give a formula for the exact sum of the alternating series, but it prescribes a way to approximate that sum with a *predetermined* level of accuracy of our choosing.

For example, the *alternating harmonic series* is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$$

Although the harmonic series diverges, the alternating harmonic series satisfies the conditions of the alternating series test and must converge. Suppose that you wanted to calculate its sum within 0.01 of the actual value.

The absolute value of the 100th term is $\frac{1}{100} = 0.01$.

Calculate the 99th partial sum

$$\sum_{n=1}^{99} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{100}}{99} + \dots$$

The error will be less than 0.01.

Use the $\boxed{2\text{nd}}$ [TABLE] feature on the graphing handheld to make this calculation by defining the appropriate sequence of partial sums in the **Y=** editor, as shown below.

```
Plot1 Plot2 Plot3
nMin=1
u(n) sum(seq((-1)^(k+1)/k, k, 1, n))
u(nMin) (1)
v(n) =
v(nMin) =
```

You can also compute it on the Home screen.

```
seq((-1)^(k+1)/k, k, 1, 99)
(1 -.5 .3333333...
Ans*Frac
(1 -1/2 1/3 -1/...
sum(Ans)
.6981721793
```

Note: It is known that the alternating harmonic series converges exactly to $\ln(2)$, the natural logarithm of 2. Compare this to three decimal places, $\ln 2 \approx 0.693$, to verify that the approximation is actually closer than 0.01 to the exact sum of the series. Keep in mind that the alternating series test establishes a “worst-case” error bound guarantee and approximations using partial sums could be much closer than this bound.

For each of the following alternating series, determine whether each meets the conditions of the Alternating Series Test. If it does, find the smallest value n such that the n th partial sum would approximate the sum of the series to within 0.01, and find the value of that partial sum.

$$6. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots + \frac{(-1)^{n+1}}{n \cdot 2^n} + \dots$$

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^{n+1}}{(n+1)!} + \dots$$

$$8. \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} = \frac{1}{2} - \frac{1}{2^3 \cdot 3!} + \frac{1}{2^5 \cdot 5!} + \dots + \frac{(-1)^n}{2^{2n+1}(2n+1)!} + \dots$$

$$9. \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}(2n)!} = 1 - \frac{1}{3^2 \cdot 2!} + \frac{1}{3^4 \cdot 4!} + \dots + \frac{(-1)^n}{3^{2n}(2n)!} + \dots$$

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \dots + \frac{(-1)^{n+1}n}{(n+1)} + \dots$$

$$11. \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{(-1)^n}{(2n+1)} + \dots$$

$$12. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2^n} = \frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \dots + \frac{(-1)^{n+1}n}{2^n} + \dots$$