

Fly Straight



Teacher Notes & Answers

7 8 9 10 11 12



Problem Statement

There are all sorts of patterns created by sequences such as the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13 ...

In this coding exercise we look at a very usual sequence with an aim to see if there are any observable patterns appear as the sequence progresses. The sequence starts with the first two terms defined: $t_0 = 1$ and $t_1 = 1$.

Term Number:	n	0	1	2
Term:	t_n	1	1	

Just like the Fibonacci sequence, this one is also recursive, meaning successive terms depend on previous terms. To calculate the next term in the sequence we check to see if there are any common factors between the highlighted values:

Term Number:	n	0	1	2
Term:	t_n	1	1	

If the highest common factor is 1 then we add the two highlighted terms then and add 1 to the result: $1 + 2 + 1 = 4$.

Term Number:	n	0	1	2	3
Term:	t_n	1	1	4	

We're ready to calculate the next term. We apply the same criteria, if the highest common factor between the two highlighted terms is 1 then we add the two highlighted terms and then add 1 to the result: $4 + 3 + 1 = 8$.

Term Number:	n	0	1	2	3	4
Term:	t_n	1	1	4	8	

We're ready to calculate the next term. Applying the same criteria we see that the highest common factor between the highlighted terms (8 and 4) is 4. When the highest common factor is not equal to 1, the term (8) is divided by the highest common factor, in this case 4. Our new term becomes: $8 \div 4 = 2$.

We'll do two more terms for practice.

Term Number:	n	0	1	2	3	4	5
Term:	t_n	1	1	4	8	2	

The highest common factor of the two highlighted terms is 1, therefore the next term is $2 + 5 + 1 = 8$.

Term Number:	n	0	1	2	3	4	5	6
Term:	t_n	1	1	4	8	2	8	

For the second time we see that the highest common factor of the two highlighted terms is not equal to 1. So the next term is equal to $8 \div 2 = 4$ (Highest common factor of 8 and 6 is 2)

Check the next two terms in the table below.

Term Number:	n	0	1	2	3	4	5	6	7	8
Term:	t_n	1	1	4	8	2	8	4	12	3

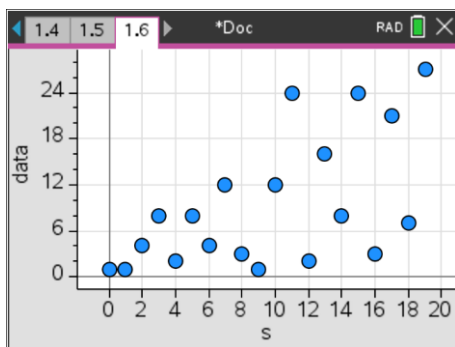
Question: 1.

Complete the next series of terms in the table. (Check the answers with your teacher before continuing)

Term Number:	n	8	9	10	11	12	13	14	15	16
Term:	t_n	3	1	12	24	2	16	8	24	3

Question: 2.

Create a scatterplot for the data collected so far with the term number (n) on the x axis (independent variable) and the term on the y axis (dependent variable). Include a copy of your scatterplot in your answers and identify any patterns if they exist.



Note: Students may draw the scatterplot by hand or populate a spreadsheet and use the calculator.

Answer: In this early scatterplot there are no obvious patterns. Students may observe that almost all the points lie under the line $y = 2x + 1$, or that prime numbers cause terms to increase in value.

Examples: $t_7 = 12$, $t_{11} = 24$ and $t_{13} = 16$.

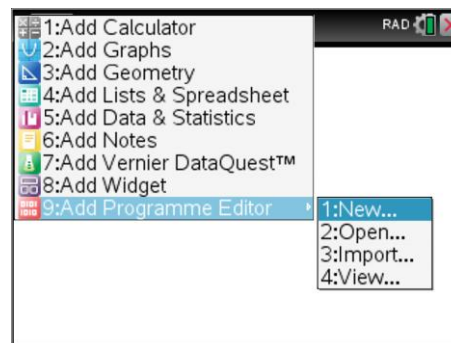
Students may also notice that the majority of the terms (at this stage) are even.

Coding the Sequence – Introduction

Start a new document and insert a Program.

Name the program: Fly

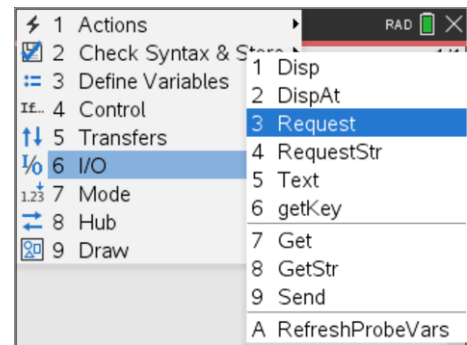
Note: You cannot have spaces or special characters in a program name.



The first step is to prompt the user for the quantity of terms to be generated by the program.

Menu > I/O > Request

Request("How many terms? ",*d*)



The term numbers can be generated by a single command:

$s := \text{Seq}(x, x, 0, d)$

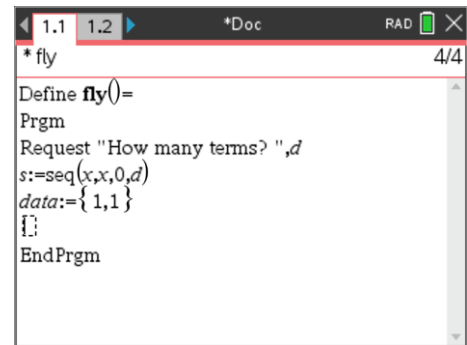
Enter the next line:

$\text{data} := \{1, 1\}$

As the sequence is generated it will be stored in a list called data.

Save and run your program (Ctrl + R) and enter the quantity of terms when prompted. (Enter a value between 1 and 100)

The program will say 'done', but what has it done?



Question: 3.

Once you have run the program, in the calculator application type: s. What does s equal and represent?

Answer: S represents the term number. {0, 1, 2, ...}

Question: 4.

In the calculator application try the following calculations:

GCD(20,30) GCD(12,16) GCD(17,12) GCD(24,12)

Explain what the GCD command does.

Answer: GCD determines the highest common factor of two numbers. (Greatest Common Divisor)

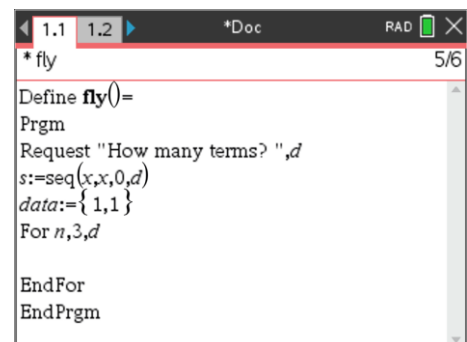
Coding the Sequence – Making Decisions

It's time to start creating the sequence. The user has provided information about the quantity of terms to be generated. (*d*)

The first two terms are already stored in data. That means a LOOP can start at the third term and finish with term '*d*'.

Menu > Control > For ... EndFor

The loop starts at 3 and ends at *d*. The loop variable is '*n*'. (See opposite)



Now for the decision process. The decision is based on whether or not the term number and the previous term have any factors in common.

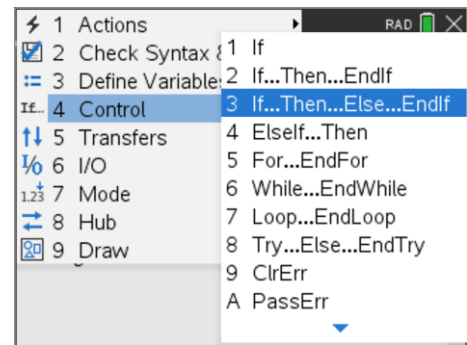
Menu > Control > IF ... THEN ... ELSE ... ENDIF

Now the condition needs to be entered.

Condition: $\text{GCD}(\text{data}[n-1], s[n]) = 1$

Note:

The square brackets are used to refer to a specific element in a list.
So $\text{data}[3] = t_3$ in mathematical terms.

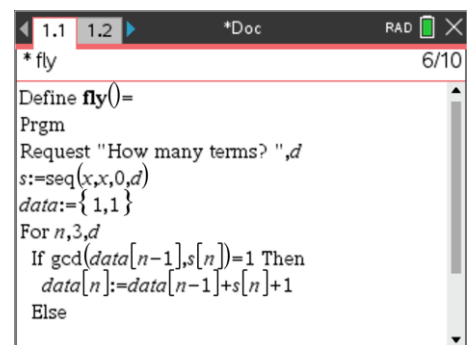


Question: 5.

Explain what the condition $\text{GCD}(\text{data}[n-1], s[n]) = 1$ is testing.

Answer: This finds the highest common factor of the n^{th} term number and the $(n - 1)^{\text{th}}$ term.

If the condition has been met the program will follow the instruction immediately following THEN

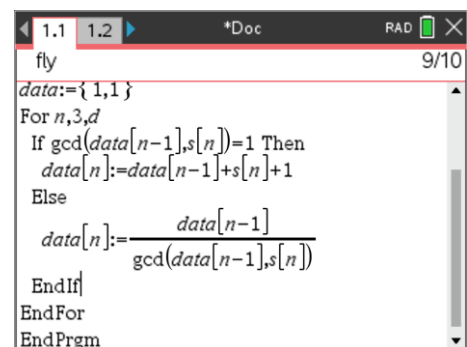
$$\text{data}[n] := \text{data}[n-1] + s[n] + 1$$


Question: 6.

Explain what this command does in reference to the sequence.

Answer: If there are no common factors the previous term, current term number and one are added together.

If the condition has not been met the program will follow the instruction immediately following ELSE

$$\text{data}[n] := \text{data}[n-1] \div \text{gcd}(\text{data}[n-1], s[n])$$


Question: 7.

Explain what this command does in reference to the sequence.

Answer: When a common factor exists between the current term number and the previous term, the previous term is divided by the highest common factor.

Mathematical Investigation - Flying Straight

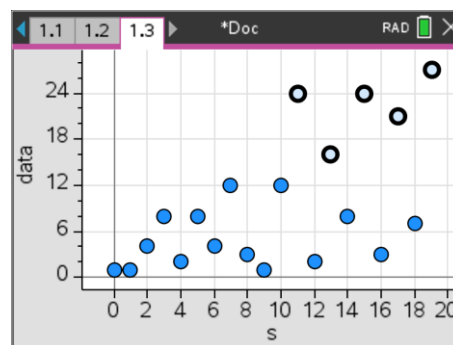
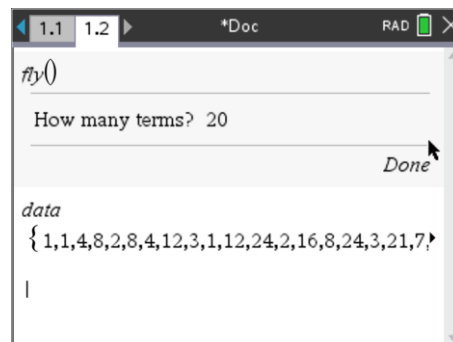
The program is now ready to run.

Pressing Ctrl + R from the programming application.

Start by entering '20' for the number of terms when prompted. Once the program says "Done", recall the Data variable to ensure the terms are correct.

Insert a Data and Statistics application and plot the term number (s) on the x axis and the term (data) on the y axis for the first 20 terms in the sequence.

The aim of this investigation is to find patterns. Five points in the first 20 terms of the sequence have been highlighted (opposite). These points sit higher than most of the other points. Given that our sequence is based on divisibility (High Common Factor), will all of these points be prime numbers?



One of the challenges faced in any mathematical investigation is that we are not sure where the investigation may lead. We can only start with clues in our initial problem construction and then start generating data. Our first clue in this investigation is that our sequence is based on divisibility. When divisibility exists (common factors), the value of the next term in the sequence decreases, the reverse is true when divisibility is not present.

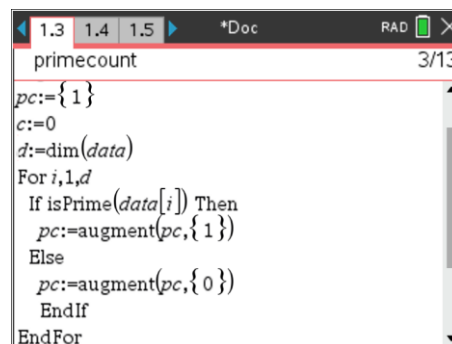
It's now time to start investigating. The program provides a tool to quickly generate lots of terms, start small. Look for clues such as:

- Terms or term numbers with a particularly high quantity of factors such as 36, 60 and 120.
- Terms or term numbers that are prime numbers.
- Relative abundance of prime and composite numbers present in the sequence. (data)
- Progressively increase the quantity of data that you are analysing. Sometimes patterns start to appear when we get to stand back from the data.

Answers & Teacher Notes

Answers for this section will vary enormously. Students should be encouraged to write program to assist with pattern searches. For example the frequency of prime numbers in the terms can be determined using a spreadsheet or by a small program (opposite). This demonstrates to students that programming can be used to assist in data analysis.

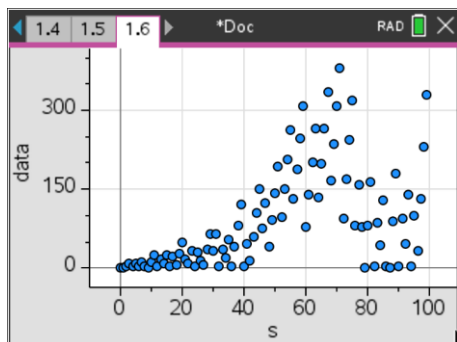
Some interesting anomalies do exist with the quantity of prime numbers in the sequence; they appear to represent approximately 25% of the terms in the sequence.



A similar investigation would be to compare the frequency of odd and even numbers in the sequence. An odd/even count program is very similar to the prime count. $\text{Mod}(\text{term}, 2) = 0$ would detect the presence of an even number. In the case of odd/even numbers the command set: `sum(mod(data,2))` will generate a count of the quantity of odd numbers.

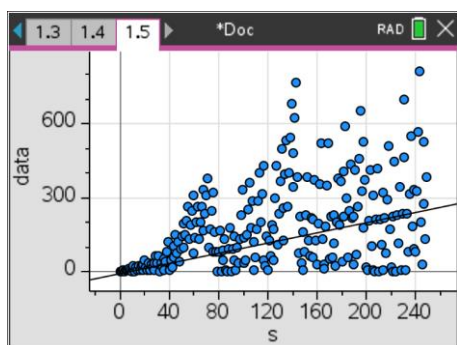
In the case of the odd/even count, for the first 2000 numbers the proportion is almost exactly 50:50.

This leads us to the title of this investigation: “Fly Straight”. The graphs below show what happens as you progressively increase the number of terms in the sequence.



First 100 Terms

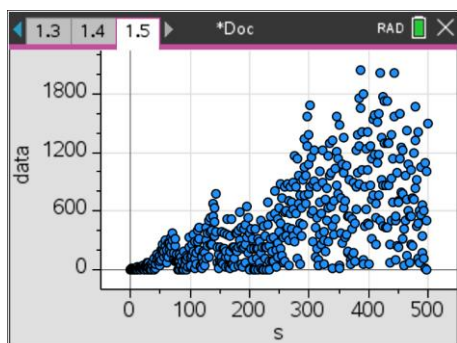
This graph represents the first 100 terms. We see that at approximately t_{70} the sequence suddenly drops and stays low for a little while. There is also a cluster of points that almost form a straight line (between 80 and 100) and also a cluster that hover around 1 or 2.



First 200 Terms

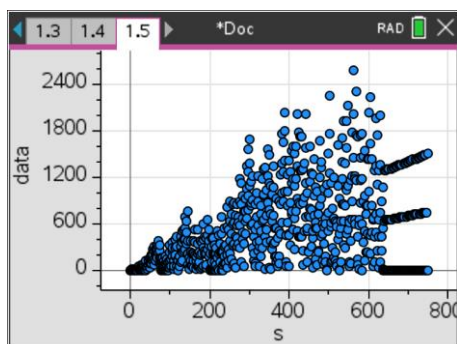
A straight line has been added to the graph to highlight that there are a number of points that appear to product a straight line. It was present in the first 100 terms and appears more pronounced in the 100 to 200 terms.

We also see that the sequence climbed relatively high as the term numbers 139, 142 and 143 appear. It is interesting to note that 139 is prime, 142 has factors $\{1, 2, 71 \text{ and } 142\}$ and 143 has factors $\{1, 11, 13, 143\}$. So the relatively low factor count means it was highlight likely that the sequence would increase.



First 500 Terms

At this point in time it appears that there will be no patterns other than the all the points appear to be staying below a line or curve (to be determined).



First 750 Terms

Okay ... so that is **REALLY** strange! (Hence the title – Fly straight). All new terms in the sequence appear to follow one of three lines. The bottom line is a collection of points oscillating around 1 and 2. The next is the line $y = x/2$ and the highest line is that of $y = x$.

What happens to this pattern going forward? Where does it start? Why?

These questions are left to the reader. The wonderful team at Numberphile (YouTube) have the answer.