The Power of Trigonometric Integrals

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Teachers Notes & Answers

7 8 9 10 11 **12**

TI-Nspire™ | Activity | Student | 4 hours

Introduction

In this activity you will:

- evaluate definite integrals involving powers of trigonometric functions,
- derive a recurrence relation,
- Define a function to verify the results.

Definite integrals involving powers of the sine function.

Question: 1.

a) Evaluate each of the following:

i) $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$ **Answer:** 1

ii) $\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$ **Answer:** $\frac{\pi}{4}$

iii) $\int_0^{\frac{\pi}{2}} \sin^3(x)\,dx$ **Answer:** $\frac{2}{3}$

iv) $\int_0^{\frac{\pi}{2}} \sin^4(x)\,dx$ **Answer:** $\frac{3\pi}{16}$

v) $\int_0^{\frac{\pi}{2}} \sin^5(x)\,dx$ **Answer:** $\frac{8}{15}$

b) A recurrence relation is an equation that recursively defines a sequence, the results in Part (a) form such a sequence.

i) Let $S(n)=\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx$ use integration by parts to show that $S(n)=\frac{n-1}{n}S(n-2)$.

General Solution:

$$\int udv = uv - \int vdu$$

$Let\ u = \sin^{n-1}(x)$ $\qquad dv = \sin(x)$

$\frac{du}{dx} = (n-1)\cos(x)\sin^{n-2}(x)$ $\qquad v = \int \sin(x)dx$

$v = -\cos(x)$

$$\int \sin^n(x)dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)dx$$

$$\int udv = -\cos(x)\sin^{n-1}(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x)dx$$

$$\int udv = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)dx - (n-1)\int \sin^2(x)dx$$

$$n\int \sin^n(x) = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)dx$$

$$\int \sin^n(x) = \frac{-1}{n}\cos(x)\sin^{n-1}(x) + \frac{(n-1)}{n}\int \sin^{n-2}(x)dx$$

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Specific Solution for the definite integral required:

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^{\frac{\pi}{2}} \sin^n(x) = \frac{-1}{n}\left[\cos(x)\sin^{n-1}(x)\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n(x) = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)dx$$

Where $S(n) = \int_0^{\frac{\pi}{2}} \sin^n(x)dx$ then it follows that: $S(n) = \frac{n-1}{n}S(n-2)$

Teacher Notes: A PowerPoint slide set is provided to step through this integration technique.

ii) Use the recurrence relation, established in the previous question, to check your answers to (a)(iii) and (a)(v).

Answer:

$$S(3) = \frac{n-1}{n}S(1) \qquad S(5) = \frac{n-1}{n}S(3)$$

$$S(3) = \frac{2}{3}(1) = \frac{2}{3} \qquad S(5) = \frac{4}{5}\left(\frac{2}{3}\right) = \frac{8}{15}$$

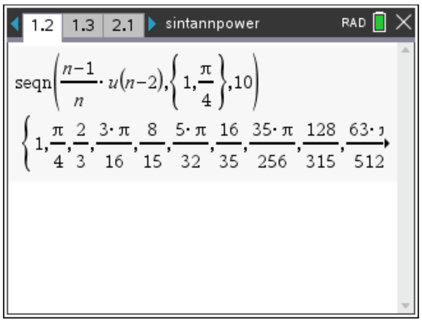
iii) Use the recurrence relation to find, $S(6), S(7), S(8), S(9), S(10)$

Answer: n = 4 and n = 5 obtained from Part (a). Remaining values determined by recursion.

n	4	5	6	7	8	9	10
S(n)	$\frac{3\pi}{16}$	$\frac{8}{15}$	$\frac{5\pi}{32}$	$\frac{16}{35}$	$\frac{35\pi}{256}$	$\frac{128}{315}$	$\frac{63\pi}{512}$

iv) Use CAS to check the values of $S(n)$ for $n = 1, 2, \ldots., 10$.

Answer: Students may use a Notes Application (exact values) or the Graph Application (Sequence), The Graph Application will generate approximate values where as the Notes Application or sequence command (seqn) work perfectly.



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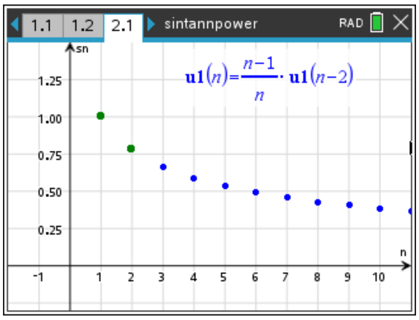
v) Graph the results for $S(n)$ versus n, for $n=1,2,....,10$.

Answer:



vi) Verify that for n even, $S(n)=\frac{\pi}{4}\prod_{j=2}^{\frac{n}{2}}\left(\frac{2j-1}{2j}\right)$ and for n odd $S(n)=\prod_{j=1}^{\frac{n-1}{2}}\left(\frac{2j}{2j+1}\right)$, and hence write a TI-Nspire function (not involving definite integrals) to evaluate $S(n)$.

Answer:

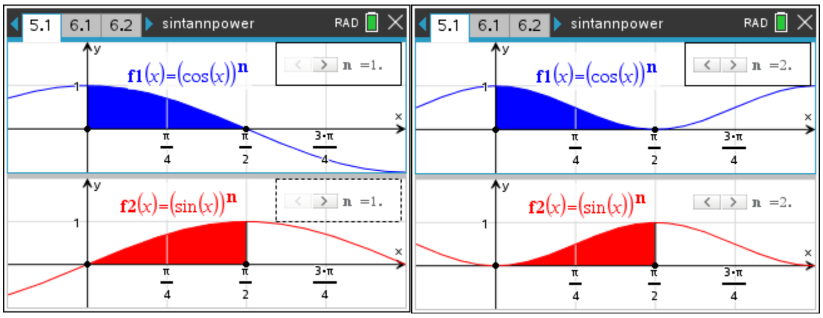
```
Define s(n)=Func
    If mod(n,2)=0 Then
        Return π/4 · ∏(j=2, n/2) ((2·j−1)/(2·j))
    Else
        Return ∏(j=1, (n−1)/2) ((2·j)/(2·j+1))
    EndIf
EndFunc
```

Definite integrals involving powers of the cosine function.

Question: 2.

a) Use graphs to help explain why $\int_0^{\frac{\pi}{2}}\cos(x)\,dx=\int_0^{\frac{\pi}{2}}\sin(x)\,dx$

Answer: The graph shows the regions are the same. This extends to $\sin^2(x)$ & $\cos^2(x)$ … $\sin^n(x)$ & $\cos^n(x)$.



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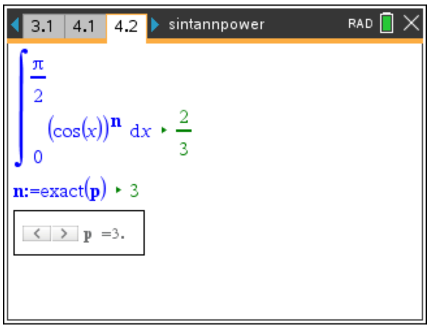
b) Let $C(n)=\int_0^{\frac{\pi}{2}}\cos^n(x)\,dx$, show that $C(n)=S(n)$ for $n=1,2,....,5.$

Answer: This example uses the Notes Application, the slider can be used to calculate each value for *n*.

c) **Answer**:



Note:
The slider generates 'approximate' values, so the integral will be approximated if the slider value is used directly. In the example shown here an additional maths-box is used: 'n:=exact(p)' so the integral will now return the exact value.

d) Show that $C(n)=S(n)$ for all $n\in Z$.

Answer: There are several ways students may 'show that …'. One option is via substition:

$$\int_0^{\frac{\pi}{2}}\cos^n(x)dx=\int_0^{\frac{\pi}{2}}\sin^n(x+\tfrac{\pi}{2})dx \qquad Let\ u=x+\tfrac{\pi}{2} \qquad upper=\pi$$

$$\frac{du}{dx}=1 \qquad lower=\tfrac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}}\cos^n(x)dx=\int_{\frac{\pi}{2}}^{\pi}\sin^n(u)du$$

$$=\int_0^{\frac{\pi}{2}}\sin^n(u)du \qquad These\ regions\ are\ the\ same.$$

Definite integrals involving powers of the tangent function.

Question: 3.

a) Evaluate each of the following:

i) $\int_0^{\frac{\pi}{4}}\tan(x)\,dx$ **Answer:** $\frac{1}{2}\log_e(2)$

ii) $\int_0^{\frac{\pi}{4}}\tan^2(x)\,dx$ **Answer:** $1-\frac{\pi}{4}$

iii) $\int_0^{\frac{\pi}{4}}\tan^3(x)\,dx$ **Answer:** $\frac{1}{2}-\frac{1}{2}\log_e(2)$

iv) $\int_0^{\frac{\pi}{4}}\tan^4(x)\,dx$ **Answer:** $\frac{\pi}{4}-\frac{2}{3}$

v) $\int_0^{\frac{\pi}{4}}\tan^5(x)\,dx$ **Answer:** $\frac{1}{2}\log_e(2)-\frac{1}{4}$

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b) Let $T(n)=\int_0^{\frac{\pi}{4}}\tan^n(x)dx$ show that $T(n)=\frac{1}{n-1}-T(n-2)$. [Do not use integration by parts]

General Solution:

$$\begin{aligned}\int\tan^n(x)dx &= \int\tan^{n-2}(x)\tan^2(x)dx\\ &= \int\left(\tan^{n-2}(x)\left(\sec^2(x)-1\right)\right)dx\\ &= \int\tan^{n-2}(x)\sec^2(x)dx-\int\tan^{n-2}(x)dx\\ &= \int u^{n-2}du-\int\tan^{n-2}(x)dx \qquad \textit{By substitution } u=\tan(x)\\ &= \frac{1}{n-1}u^{n-1}-\int\tan^{n-2}(x)dx\\ &= \frac{1}{n-1}\tan^{n-1}(x)-\int\tan^{n-2}(x)dx\end{aligned}$$

Specific Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\tan^n(x)dx &= \left[\frac{1}{n-1}\tan^{n-1}(x)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan^{n-2}(x)dx\\ &= \frac{1}{n-1}-\int_0^{\frac{\pi}{4}}\tan^{n-2}(x)dx\end{aligned}$$

$$\textit{Let } T(n)=\int_0^{\frac{\pi}{4}}\tan^n(x)$$

$$T(n)=\frac{1}{n-1}-T(n-2)$$

c) Use the recurrence relation obtained in the previous question to check your answers to Q3(a).

Answer: Initial terms required $T(1)=\frac{1}{2}\log_e(2)$ and $T(2)=1-\frac{\pi}{4}$

Using the recurrence relation:

$$T(3)=\frac{1}{2}-\frac{1}{2}\log_e(2) \qquad T(4)=\frac{1}{3}-\left(1-\frac{\pi}{4}\right)=\frac{\pi}{4}-\frac{2}{3}$$

$$T(5)=\frac{1}{4}-\left(\frac{1}{2}-\frac{1}{2}\log_e(2)\right)=\frac{1}{2}\log_e(2)-\frac{1}{4}$$

d) Use the recurrence relation to find: $T(6),T(7),T(8),T(9),T(10),T(11),T(12),T(13)$.

Answer:

$$T(6)=\frac{13}{15}-\frac{\pi}{4}, \qquad T(7)=\frac{5}{12}-\frac{1}{2}\log_e(2), \qquad T(8)=\frac{\pi}{4}-\frac{76}{105},$$

$$T(9)=\frac{1}{2}\log_e(2)-\frac{7}{24}, \qquad T(10)=\frac{263}{315}-\frac{\pi}{4}, \qquad T(11)=\frac{47}{120}-\frac{1}{2}\log_e(2),$$

$$T(12)=\frac{\pi}{4}-\frac{2578}{3465}, \qquad T(13)=\frac{1}{2}\log_e(2)-\frac{37}{120}$$

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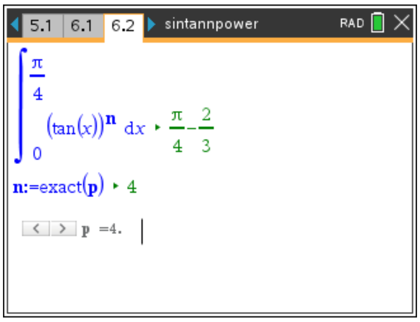
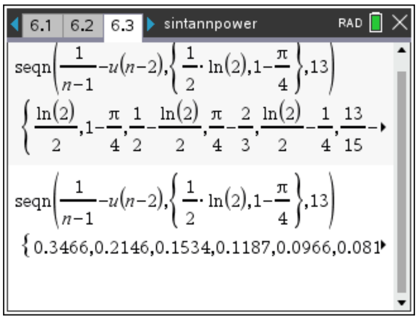
e) Use CAS to check the values of $T(n)$ for $n = 1, 2, ..., 13$.

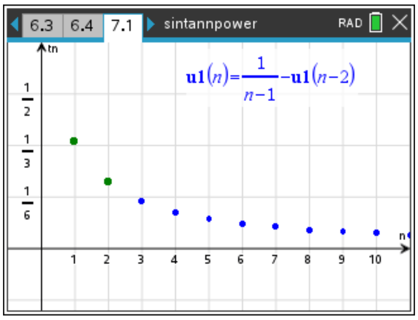
Answer: The previous Notes Application can be editted by changing the function and terminals accordingly.



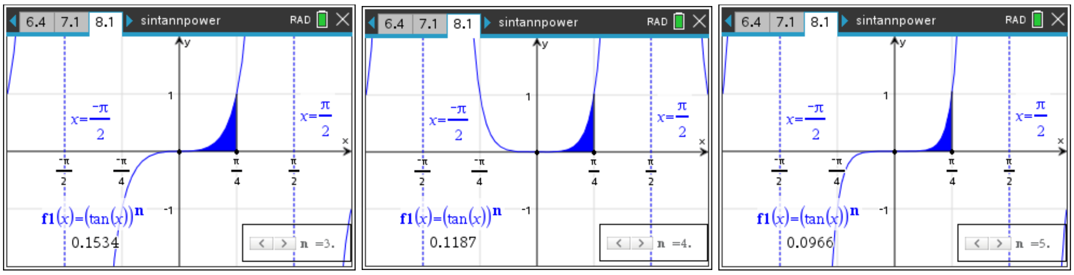


f) Graph the results for $T(n)$ versus *n*, for $n = 1, 2, ..., 10$.

Answer:



Plotting the function and using a slider helps see why the results alternate, but still are decreasing.



g) Define the function shown here and use it to verify that

when *n* is divisible by 4: $T(n) = \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{(-1)^{k+1}}{2k+1}\right) + (-1)^{\frac{n}{2}}\frac{\pi}{4}$,

when *n* even and not divisible by 4: $T(n) = \sum_{k=0}^{\frac{n}{2}-1}\left(\frac{(-1)^{k}}{2k+1}\right) + (-1)^{\frac{n}{2}}\frac{\pi}{4}$,

and, when *n* is odd: $T(n) = (-1)^{\frac{n-1}{2}}\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{(-1)^{k}}{2k}\right) + (-1)^{\frac{n-1}{2}}\frac{1}{2}\log_e(2)$.

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Answer:

Define $t(n)$=Func

If $\text{mod}(n,4)=0$ Then

Return $\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\left(\frac{(-1)^{k+1}}{2\cdot k+1}\right)+\frac{(-1)^{\frac{n}{2}}\cdot\pi}{4}$

EndIf

If $\text{mod}(n,2)=0$ and $\text{mod}(n,4)\neq 0$ Then

Return $\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\left(\frac{(-1)^{k}}{2\cdot k+1}\right)+\frac{(-1)^{\frac{n}{2}}\cdot\pi}{4}$

EndIf

If $\text{mod}(n,2)=1$ Then

Return $\displaystyle(-1)^{\frac{n-1}{2}}\cdot\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{(-1)^{k}}{2\cdot k}\right)+\frac{(-1)^{\frac{n-1}{2}}\cdot 1}{2}\cdot\ln(2)$

EndIf

EndFunc

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