Towering Mathematics

Teacher Notes

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TI-Nspire CAS | Investigation | Student | 90min

Aim:

To determine a relationship between the number of disks in the 'Tower of Hanoi' and the number of moves required to transfer the disks from one stack to another.

Number & Algebra – Year 9: Linear and non-linear relationships

Graph simple non-linear relations with ~~and without~~ the use of digital technologies and solve simple related equations (ACMNA296)

ScOT: Linear functions

Number & Algebra – Year 10: Linear and non-linear relationships

Explore the connection between algebraic and graphical representations of relations ~~such as simple quadratics, circles and exponentials~~ using digital technology as appropriate (ACMNA239)

ScOT: Exponentials, Data Representation

Equipment:

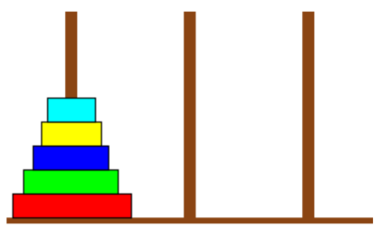
For this activity you will need:

- TI-Nspire CAS
- TI-Nspire CAS file (tns): HanoiV2

Students can use a physical models of this problem for initial exploration instead of this digital resource, however the act of counting and tracking moves can distract from the observation of pattern. The digital model allows students to focus on the pattern whilst accurately recording the number of moves. Feedback is also provided with regards to whether the problem could be solved in fewer moves.

Problem Description:

The Tower of Hanoi problem, also called Tower of Brahma was invented by the French Mathematician, François Édouard Anatole Lucas in 1883. The objective is simple:



Move all the disks from one column to another.

The rules:

- Disks must always be stacked on a column (spike)
- Move only one disk at a time
- A large disk can not be placed on a small disk

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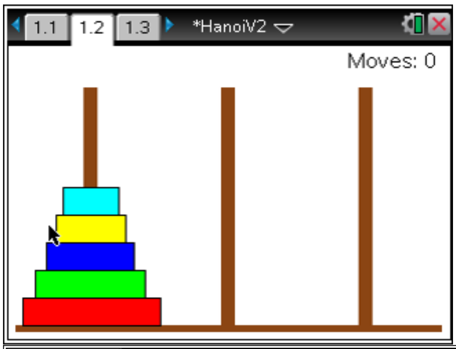
The aim of this investigation is to develop a rule relating the *number of disks* (x) and the *number of moves* (y) required to solve the problem.

Technology:

Open the file:

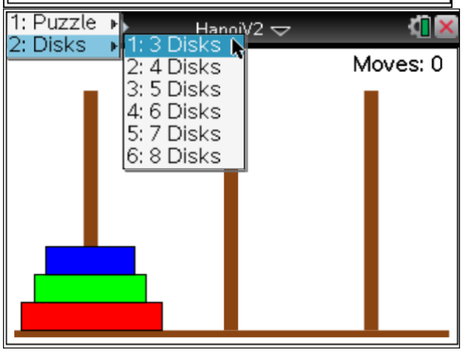
HanoiV2.tns

Navigate to the problem page (shown opposite).



Setting the number of disks

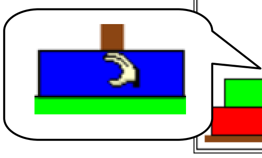
Use the [**menu**] key and set the number of disks to three.

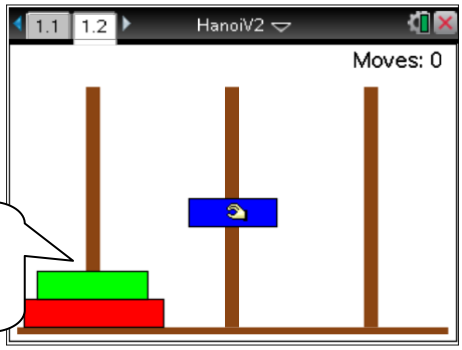


- The single and two disk problems will be determined by reasoning rather than modelling.

Moving Disks

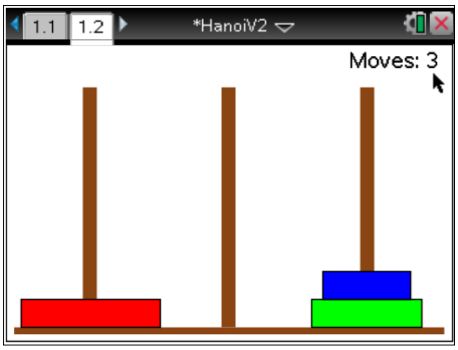
Move the mouse over the top disk, the mouse changes to an open hand. Press and hold the touchpad to grab the disk. Move the disk to a new column, press and hold the touchpad to release the disk.





Moves

The number of moves is automatically recorded in the top right corner of the screen.



- Once a move has been made, you can't take it back.

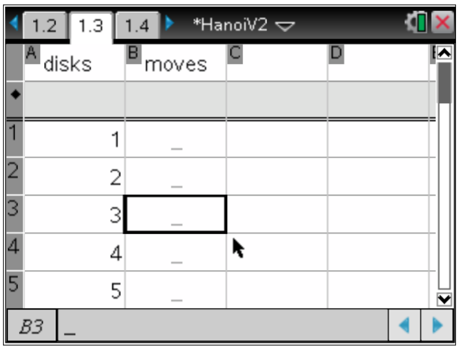
Solve the three disk puzzle and make a note of the number of moves.

Entering Data

Navigate to the spreadsheet on page 1.3

Enter the numbers 1 to 8 in column A.

Leave the first two entries in column B blank. These will be included by reasoning and logic later. Record the minimum number of moves used to solve the three disk problem in cell B3.



Students use a spreadsheet to tabulate results. If students have a reasonable working knowledge of spreadsheets they should be encouraged to develop a recursive formula for the rule relating the number of disks and the number of moves required to solve this problem.

Questions

1. Use the puzzle to determine and record the minimum number of moves required to solve:

    a. The three disk problem. 7 moves

    b. The four disk problem. 15 moves

If the number of moves increases by the same amount each time a disk is added the relationship is said to be linear.

2. If the relationship between disks and moves is linear, how many moves should it take to solve the five disk problem?
Three disk problem = 7. Four disk problem = 15. Increased by 8 moves. If the relationship is linear then the five disk problem will take 23 moves. (15 + 8). This question combined with the linearity definition (above is aimed at focusing students on number patterns. Students should be able to recognise linear and non-linear number patterns, not just their visual and graphical representation.

3. Use the puzzle to determine the minimum number of moves required to solve the five disk problem and hence determine if the relationship is linear or non-linear.
Record your answer to the five disk problem in the spreadsheet.
The purpose of this question in conjunction with the previous question is to have students understand that the problem is not linear. The linear model does not provide an accurate (correct) prediction for the number of moves. More subtly, this question looks at why equations and models need to be accurate. An appropriate and relative discussion about the importance of accurate modelling can be included at this point. Examples such as modelling global warming patterns. Mathematicians and scientists have developed models to try and predict long term weather patterns. Can we accurately predict long term forecasting (50yrs) when we can not accurately predict the weather for the next day?

4. Determine the number of moves required to solve the one and two disk problems. Explain how you determined these values.

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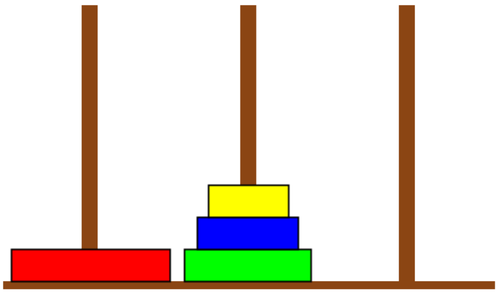
Record the number of moves in the spreadsheet.
This data provides further evidence that the model is not linear.

5. Navigate to the graph page, with your data for the one, two ... and five disk problems in the spreadsheet you will see five points on the graph. Do the points on the graph form a straight line or a curve?
   Students will observe from the graph that the points do not lie in a straight line. While the graph provides a visual representation of this non-linear relationship, it is more abstract than the numerical data.

6. The image shown opposite is a partial solution to the four disk problem. The large disk has yet to be moved.

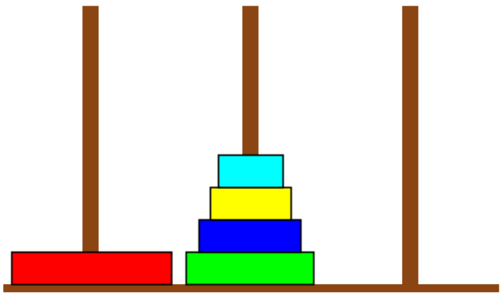


   a. How many moves have taken place so far?
      The three disk problem has essentially been solved. This scaffolding aims to help students break the problem down by using previous answers to help identify the relationship. (7 moves)

   b. Without referring to your previous data, how many moves are required to complete the puzzle?
      The large disk needs to be moved (one move) then the three disk problem needs to be solved again (seven moves).

   c. Explain how you determined your answer to the previous question.
      Double the number of moves for the three disk problem plus one move for the bottom disk. (Development of recursive formula)

7. The image shown opposite is a partial solution to the five disk problem. The large disk has yet to be moved.



   a. How many moves have taken place so far?
      15 moves

   b. How many moves are required to complete the puzzle?
      15 + 1 = 16 moves

8. Use logic and your answers to the previous questions to predict the number of moves required to solve the six disk problem.
   31 moves for previous problem, therefore total moves for 6 disks: 31 + 1 + 31 = 63

9. Set the number of disks in the software to six and solve the problem; record your answer in the spreadsheet.

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10. Compare your prediction and practical solution in the previous two questions and explain any patterns you have observed relating the number of disks and moves.
    Double the previous answer and add one.

11. Predict the values for the seven and eight disk problems.
    Enter the predicted values in the spreadsheet.
    Students may use the spreadsheet to determine the number of moves for the 7 and 8 disk problems or they may use pen and paper techniques. It is not expected at this level that students will independently devise an exponential equation. The questions in the 'developing a rule' section scaffold the rule development.
    Seven disk problem: 127
    Eight disk problem: 255

Developing a Rule

The aim of this section is to develop a mathematical rule or formula for predicting the number of moves based on the number of disks in the problem.

12. Find the next 4 terms in the following sequence: 2, 4, 8, 16, ___, ___, ___, ___.

    Sequence: 2, 4, 8, 16, 32, 64, 128, 256

13. Compare the sequence in the previous question with the number of moves required to solve the one, two, three... and eight disk problems.

    Sequence from Tower of Hanoi: 1, 3, 7, 15, 31, 63, 127, 255

    The Tower of Hanoi moves are one less than the sequence in the previous question.

14. Navigate to the calculator page (1.5) and press the [**menu**] key, from the **number** menu select **factor**. Write the result for each of the following:

    a. Factor(4) Factor(4) = 2^2

    b. Factor(8) Factor(8) = 2^3

    c. Factor(16) Factor(16) = 2^4

    d. Factor(32) Factor(32) = 2^5

15. Use your answers to the previous three questions to formulate a rule relating the number of disks (x) and the number of moves (y) to solve the Tower of Hanoi problem.

Number of disks (x)	1	2	3	4	5	6	7	8
Number of moves (y)	1	3	7	15	31	63	127	255
Expression: 2^x	2	4	8	16	32	64	128	256
Rule: $y = 2^n - 1$	1	3	7	15	31	63	127	255

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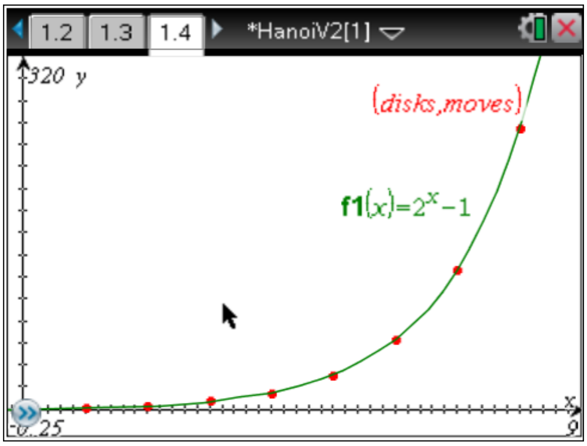
16. Check your formula from the previous question by drawing a graph on page (1.4).
    Note: You will need to change the graph type first. Press [**menu**] – **Graph Type** – **Function**.



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