**Concepts**

The general method used to construct a slope field can be used to determine a numerical approximation to the solution of a differential equation. Euler’s method is based on the idea of local linearity, that is, a differentiable function is essentially linear on small intervals. This method can be used to produce a set of straight-line segments that approximates the graph of the solution to the differential equation, and to provide a numerical approximation to a point on the solution curve.

Suppose we know the value of a function and its derivative at a single point. We can use this information to approximate a small portion of the graph of  using a straight-line segment; the tangent line to the graph of  at that point.

Consider a differential equation and an initial condition: , . The objective is to find approximate points on the solution curve at equally spaced numbers

 where  is the step size. The differential equation is used to find the slope of the tangent line at each point, for example, the slope at  is .

The approximate value of the solution to the differential equation when  is



The approximate value of the solution to the differential equation when  is



And, in general



**Course and Exam Description Unit**

Section 7.5: Approximating Solutions Using Euler’s Method

**Calculator Files**

EulersMethod.tns

**Using the Document**

EulersMethod.tns: On page 1.2, the derivative  is defined in a Math Box. The default definition for  is  . This expression can be changed by the user to allow for more in-depth and conceptual questions concerning Euler’s Method. The initial condition, the endpoint

-value, and the number of Euler steps are also defined on page 1.2.

Page 1.3 is a Lists and Spreadsheet page that displays , , and . Page 1.4 shows a graph of the points obtained using Euler’s Method. The slider for  is used to change the number of steps and the slider for  is used to step through each Euler approximation

Page 1.1

|  |  |
| --- | --- |
|  | This page provides the notation use in this calculator file associated with Euler’s Method. The initial value problem is , . Euler’s Method is used to approximate the value of the solution  at  using  steps of equal size. |

Page 1.2

|  |  |
| --- | --- |
|  | The derivative, given by the function , is defined in a Math Box. Note that this expression can be a function of both the variables  and , and can be changed by the user to allow further exploration. The initial condition is also specified on this page, in two separate Math Boxes. And the value for the endpoint, , is also specified here, in a Math Box. There is a slider used to set the number of Euler steps. The value of  is automatically computed and displayed. |

Page 1.3

|  |  |
| --- | --- |
|  | This Lists and Spreadsheet page displays , , and  for the current differential equation and initial value, endpoint , and number of steps . Click in any cell to see a more accurate value in the entry line at the bottom of the screen. |

Page 1.4

|  |  |
| --- | --- |
|  | This page is a visualization of Euler’s Method. Each step in the approximation procedure is plotted on the graph. The value of  (the number of steps) can be changed on this page. The variable  represents the step number. Use the slider to step through the solution. The coordinates of the th step are displayed on the graph screen. Here, we can see that the approximation for  in this example is . |

**Suggested Applications and Extensions**

Use the default initial value problem, , , to answer questions 1-3. The values for , , , and  can be set either in a Math Box or by using a slider. The default values are , , , and . The numerical approximations are given on page 1.3, a Lists and Spreadsheet page, and a visualization of the approximation is given on page 1.4.

1. Use Euler’s Method to approximate  for each of the following values for : (i) , (ii) , (iii) . Which value of  do you think produces the best estimate for ? Why?
2. Use Euler’s Method to approximate  for each of the following values for : (i) ,

 (ii) , (iii) . Which value of  do you think produces the best estimate for ? Why?

1. Use Euler’s Method to approximate  for . Use separation of variables to find an expression for  in terms of . Add the graph of  on page 1.4 and compare it to approximation produced by Euler’s Method. Use the graph of  to explain why the Euler approximation for  is an underestimate of the true value for .

**Additional Problems**

1. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , .
2. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , . Consider each step in this Euler approximation. Explain why the estimate for  is so much larger than the estimate for .
3. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , . Find  in terms of  and , and use this expression to explain why this approximation is an underestimate or an overestimate for the true value of .
4. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , . Use separation of variables to find an expression for  in terms of . Graph  and the Euler approximation on the same coordinate axes. Explain why the first few Euler approximations are below the graph of  and the remaining approximations are above the graph of .
5. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , . Use  to estimate . Which estimate do you think is better? Why?
6. Use Euler’s Method with  to estimate  where  is the solution to the initial-value problem , . Use separation of variables to find an expression for  in terms of . Graph  and the Euler approximation on the same coordinate axes. Find  and use this to explain why the Euler approximation for  is an underestimate of the true value for .
7. Let the function  be the solution to the differential equation  such that .
8. The function  has a critical point at . What is the -coordinate of this critical point?
9. Find  in terms of  and . Use  to determine whether the critical point found in part (a) is a relative minimum, relative maximum, or neither. Justify your answer.
10. The function  has an inflection point at . Use Euler’s Method with  to estimate  where . Is this approximation an overestimate or an underestimate. Justify your answer.