Integration by Parts

ID: 9851

Time required
45 minutes

Activity Overview

In previous activities, students have explored the differential calculus through investigations of the methods of first principles, the product and quotient rules. In this activity, the product rule becomes the basis for an integration method for more difficult integrals.

Topic: Techniques of Integration

- *Derive the formula for integration by parts and use it to compute integrals*

Teacher Preparation and Notes

- *This activity can serve to consolidate earlier work on the product rule and on methods of integration. It offers a suitable introduction to the method of integration by parts.*
- *Begin by discussing approaches to more difficult integrals and review the methods used for differentiation. It is advisable that students have some experience with substitution methods for integration before attempting this activity.*
- *Students can sit in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Most of the solutions presented below are in the TI-Nspire file and can be found by pressing* menu*, Check Answer on question pages.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student and solution TI-Nspire documents (.tns files) and student worksheet, go to education.ti.com/exchange and enter "9851" in the keyword search box.***

Associated Materials

- *IntegrationByParts_Student.doc*
- *IntegrationByParts.tns*
- *IntegrationByParts_Soln.tns*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

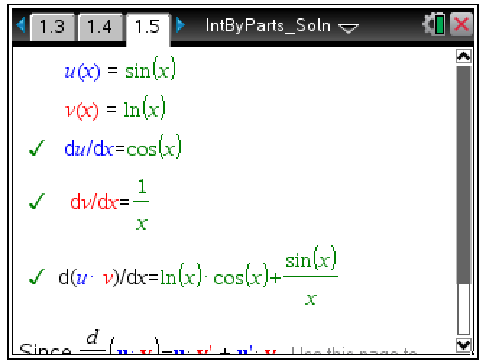
- *Exploring the Area Under a Curve (TI-Nspire Technology) — 8268*
- *Area Accumulation (TI-Nspire Technology) — 12241*

Problem 1 – Using the Product Rule

Begin with a discussion concerning integration methods and review standard integrals. Some review of substitution methods of integration is recommended. Students should be aware that not all functions can be integrated symbolically.

Step 1: Students are to apply the product rule of differentiation to the function $f(x) = \sin(x) \cdot \ln(x)$.

Review the product rule and the process of integration (particularly drawing distinction between the integral of a function and the area under the curve of the graph of that function) if needed.

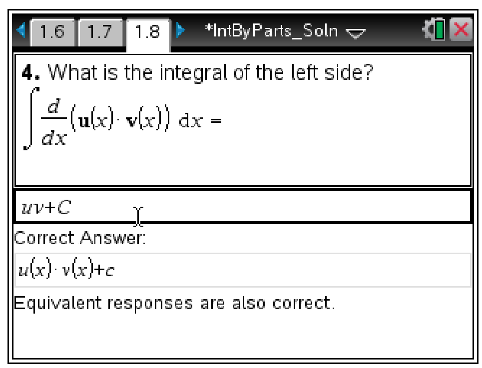


TI-Nspire™ Navigator™ Opportunity: *Quick Poll*

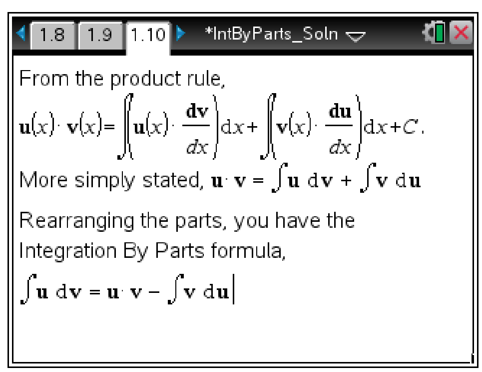
See Note 1 at the end of this lesson.

Step 2: On page 1.7, students are to take the integral of the left and right side of the product rule in general terms.

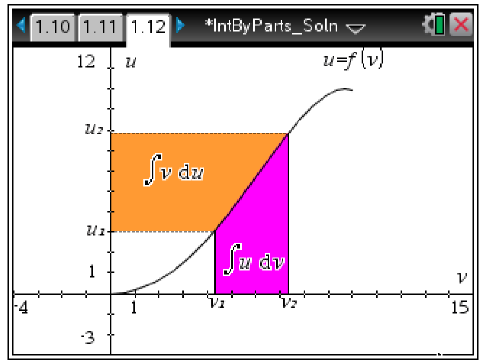
<u>**Note**</u>**:** Students should be aware that integrals evaluated by CAS generally do not include the constant term.



Step 3: Integrating the product rule and rearranging the parts provides an expression which proves to be the basis for the method of integration by parts. However, it is recommended that this process is not simply *<u>assumed</u>* as valid, but that students should be given the opportunity to *<u>engage</u>* with it, to discuss and to challenge.

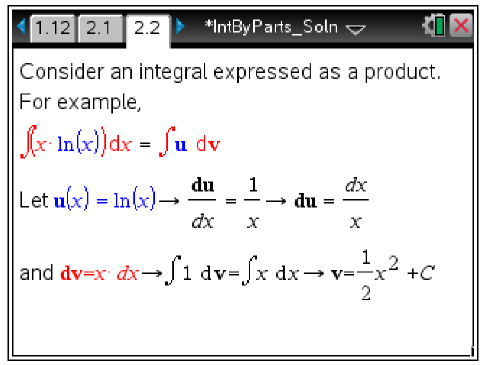


Step 4: A graphical approach can help students to appreciate the significance of each of the component parts of the integration by parts statement: the areas under the curve **with respect to *u*** and **with respect to *v*** are key concepts here. Interestingly, it is the product, ***u*v*** which is most difficult to draw from this scenario. It represents a shorthand for the difference product, $u_2 \cdot v_2 - u_1 \cdot v_1$. This is shown on the diagram as the L-shaped area between the curve and the axes, between the given limits.



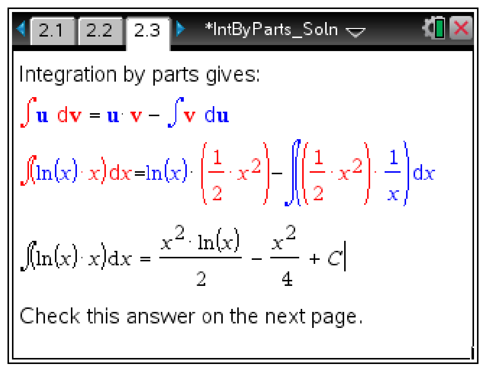
Problem 2 – Examples of Integration by Parts

Step 1: Drawing together these observations, students should be assisted to see the significance of this method, which involves correctly re-stating the function in question in order to take the derivative of one component and the integral of the other.

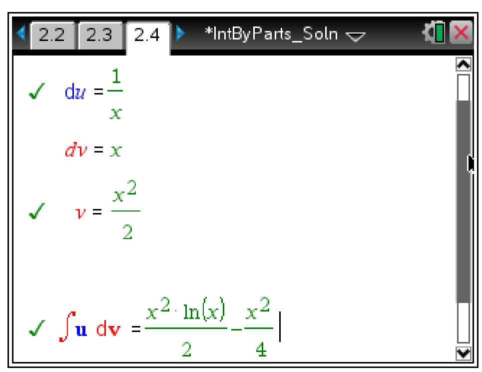


Step 2: Several worked examples are provided, beginning with the relatively straight-forward ***x*ln(*x*)**. The challenge soon emerges: *how to identify which part of the product becomes* ***u*(*x*)** *and which part becomes* ***dv***.

Whether this choice actually makes any difference to the result becomes the basis for an extension exercise later.



Step 3: The ability to evaluate certain more-difficult integrals which do NOT appear to be products provides the basis for the next key worked example. In the case of the natural logarithm, we do not at present know the integral for this function, but we do know its derivative, and can easily integrate 1. This opens the doors as an approach for a wide range of challenging integration questions.

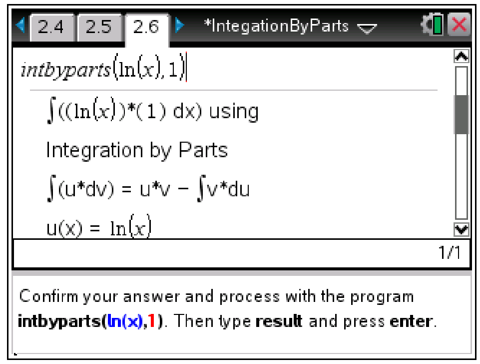


TI-Nspire™ Navigator™ Opportunity: *Class Capture*

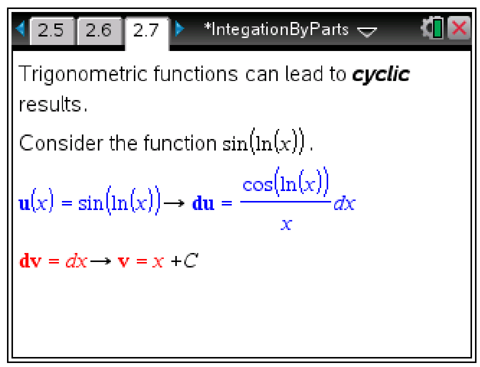
See Note 2 at the end of this lesson.

Step 4: As with the preceding CAS-based Calculus activities, a prepared program, in this case, **intbyparts(fn1, fn2)**, is available for students to verify their own work, check their conjectures and results, and to serve as a model for a complete worked solution to such questions.

Note that the two input components are derived simply from the given function—the first is defined as ***u*(*x*)** and the second as ***dv/dx***. The final step in the program is left to the students, who can check their answer by typing the variable **result**.

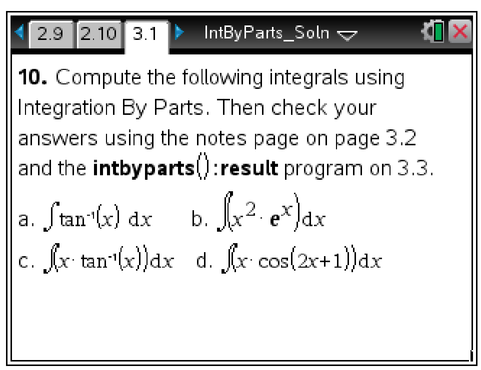


Step 5: The third worked example involves trigonometric functions, which often lead to cyclic results—an integral involving a ***sine*** will generally lead to another integral involving ***cosine***. Applying the rule a second time will often result in another ***sine*** result, which may be then set as an equation involving the original integral and solved accordingly. Reduction rules may also be developed in this way, leading to a method for integrating trig functions of higher powers.



Problem 3 – Practice using Integration by Parts

Practice questions are provided and then two extension challenges: does the order of choice make any difference (sometimes), and is there an integration equivalent to the quotient rule (no, too difficult to establish).



Solutions

1. $\frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}$
2. $\frac{d}{dx}\left(\sin(x)\cdot\ln(x)\right)=\sin(x)\cdot\frac{1}{x}+\ln(x)\cdot\cos(x)$
3. No; While $\frac{d}{dx}(\int f(x)dx)=f(x)$ is true, $\int\frac{d}{dx}(f(x))dx=f(x)+C$
4. $\int\left(\frac{d}{dx}\left(u(x)\cdot v(x)\right)\right)dx=u(x)\cdot v(x)+C$
5. $v(x)\cdot\int u(x)dx+u(x)\cdot\int v(x)dx$
6. The area between the curve ***u = f(v)*** and the ***v***-axis (between the specified limits, i.e. $\int_{v_1}^{v_2}u\cdot dv$) can be found by taking the area of the large rectangle $(u_2\cdot v_2)$ minus the smaller rectangle $(u_1\cdot v_1)$, and then subtracting the area of the region between the curve and ***u***-axis, between the specified limits ($\int_{u_1}^{u_2}v\cdot du$).
7. $\int\left(\ln(x)\cdot 1dx\right)$ using integration by parts

   $u(x)=\ln(x)\qquad du=\frac{1}{x}dx$

   $dv=1dx\qquad v=x$

   $\int\ln(x)\cdot 1dx=\ln(x)\cdot x-\int x\cdot\frac{1}{x}dx=x\cdot\ln(x)-x$
8. $\int\cos\left(\ln(x)\right)dx=x\cdot\cos\left(\ln(x)\right)+\int\sin\left(\ln(x)\right)dx$
9. $\int\sin\left(\ln(x)\right)\cdot 1dx=x\sin\left(\ln(x)\right)-\left[x\cos\left(\ln(x)\right)+\int\sin\left(\ln(x)\right)dx\right]$

   $=\frac{x}{2}\left[\sin\left(\ln(x)\right)-\cos\left(\ln(x)\right)\right]$
10. a) $\int\tan^{-1}(x)\cdot 1dx$ using integration by parts

   $u(x)=\tan^{-1}(x)\qquad du=\frac{1}{x^2+1}dx$

   $dv=1dx\qquad v(x)=x$

   $\int\tan^{-1}(x)\cdot 1dx=\tan^{-1}(x)\cdot x-\int x\cdot\frac{1}{x^2+1}dx$

   $=x\tan^{-1}(x)-\frac{1}{2}\ln(x^2+1)+C$

b) $\int x^2 \cdot e^x dx$ using integration by parts

$u(x) = x^2 \qquad du = 2x$

$dv = e^x dx \qquad v(x) = e^x$

$\int x^2 \cdot e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x\, dx$

Repeat integration by parts for $\int e^x \cdot 2x\, dx$

$u(x) = 2x \qquad du = 2\, dx$

$dv = e^x dx \qquad v(x) = e^x$

$\int e^x \cdot 2x\, dx = 2x \cdot e^x - 2\int e^x dx = 2x \cdot e^x - 2e^x$

Hence, $\int x^2 \cdot e^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x = e^x(x^2 - 2x + 2)$

c) $\int x \tan^{-1}(x) dx$ using integration by parts

$u(x) = \tan^{-1}(x) \qquad du = \dfrac{1}{x^2+1} dx$

$dv = x\, dx \qquad v(x) = \dfrac{x^2}{2}$

$$\begin{aligned}\int x \tan^{-1}(x) &= \tan^{-1}(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1} dx \\ &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2}\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} dx \\ &= \frac{x^2 \tan^{-1}(x)}{2} - \frac{1}{2}\left(x - \tan^{-1}(x)\right) \\ &= \frac{1}{2}\left((x^2+1)\tan^{-1}(x) - x\right)\end{aligned}$$

d) $\int x \cos(2x+1) dx$ using integration by parts

$u(x) = x \qquad du = 1\, dx$

$dv = \cos(2x+1)\, dx \qquad v(x) = \dfrac{\sin(2x+1)}{2}$

$$\begin{aligned}\int x \cos(2x+1) &= x \cdot \frac{\sin(2x+1)}{2} - \int \frac{\sin(2x+1)}{2} \cdot 1\, dx \\ &= \frac{x \sin(2x+1)}{2} + \frac{\cos(2x+1)}{4} \\ &= \frac{1}{4}\left(2x \sin(2x+1) + \cos(2x+1)\right)\end{aligned}$$

11. Often it does not matter, if both functions are readily differentiable and integrable, but in cases where one may not be reduced by the process, it can make a difference.

12. No, since the quotient rule formula does not allow us to easily rearrange the parts—the denominator v^2 causes problems.

TI-Nspire™ Navigator™ Opportunities

Note 1
Problem 1, *Quick Poll*
This would be a good place to do a quick poll to verify that students understand each of the parts. You may choose to ask questions about the values they are entering into the table, such as "*what is u?"* or "*what is dv?"* Quick polls can be given throughout the activity to ensure students are following along and to elicit discussion. The questions on the worksheet may be used as a guide. Any question page that follows would also be a good spot to do a Quick Poll.

Note 2
Problem 2, *Class Capture*
You may want to use Class Capture throughout the activity to verify students are entering the correct data and working successfully through each problem.