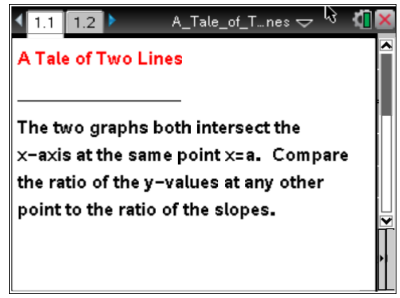
A Tale of Two Lines



Math Objectives

- Students will be able to determine limits of ratios of functions appearing linear using approximation.
- Students will recognize the relationship between the ratio of slopes of linear functions and the ratios of the values of linear functions.
- Students will be able to apply the preceding ideas to non-linear functions by recognizing the relationships between local linearity, slopes of functions, and the derivatives of functions.
- Students will learn and apply l'Hôpital's Rule.
- Students will make sense of problems and persevere in solving them. (CCSS Mathematical Practice)
- Students will construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)

Vocabulary

- limit
- derivative
- differentiable

About the Lesson

- This lesson involves demonstrating a visual justification for l'Hôpital's Rule as applied to $\frac{0}{0}$ forms.
- As a result, students will:
  - Begin with a zoomed-in graph of two functions, displaying both functions as linear. They will observe that the ratio of the slopes of the functions is the same as the ratio of the *y*-values of the function near the point where both are 0.
  - Zoom out on the functions, revealing two non-linear functions. They will note that the limit of the quotients of the functions at their point of intersection cannot be determined algebraically.
  - Recognize that the slope of the zoomed-in functions is the same as the derivative of the functions at that point, and use that information to justify l'Hôpital's Rule.

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- Use Quick Poll to assess student understanding.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

- A_Tale_of_Two_Lines_Student.pdf
- A_Tale_of_Two_Lines_Student.doc

TI-Nspire document

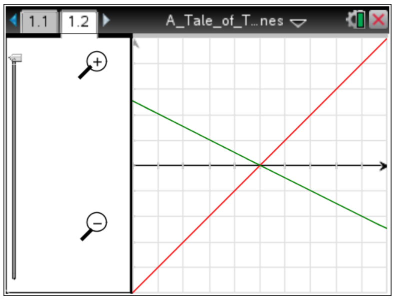
- A_Tale_of_Two_Lines.tns

Discussion Points and Possible Answers

Tech Tip: Click the slider in order to use the keypad arrows (▲ and ▼) to arrow up and down.

Tech Tip: Drag the slider up to zoom in on the graph and down to zoom out.

Move to page 1.2.



1. Note that the grid markings on the graph represent the same scale for both *x* and *y*. These appear to be the graphs of two linear functions. Call the function with positive slope $\mathbf{f}(x)$ and the function with negative slope $\mathbf{g}(x)$.

   a. What is the slope of the graph of **f?** What is the slope of the graph of **g?** How do you know?

   <u>**Answer:**</u> The slope of the graph of **f** is 1 and the slope of the graph of **g** is $-\frac{1}{2}$. They can be found visually from the graph, by observing that the graph of **f** goes up one unit and over one unit, while the graph of **g** goes down one unit and over 2 units.

   b. Suppose the intersection point of the two graphs is (*a*, 0). Use the slope information from part 1a and the point slope form to find expressions for $\mathbf{f}(x)$ and for $\mathbf{g}(x)$ that would fit these two linear graphs.

   <u>**Answer:**</u> The equation for **f** is $\mathbf{f}(x) = x - a$, and the equation for **g** is $\mathbf{g}(x) = -(1/2)(x - a)$.

2. Again, note that the grid markings on the graph represent the same scale for both *x* and *y*.

   a. Use the grid to find the ratio $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ for 4 values of *x* that are different than $x = a$.

   <u>**Answer:**</u> All the ratios should be –2.

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See Note 1 at the end of this lesson.

b. Based on your answer to part 2a, what do you think the limit of $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ is as x approaches *a?*

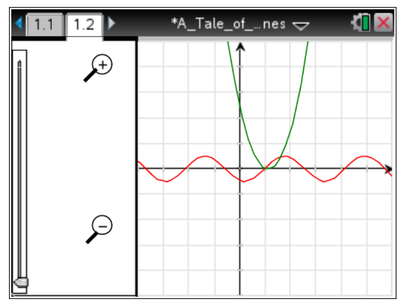
Explain your reasoning.

Answer: The limit is –2. Students' reasoning will vary, but they will likely note that $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ approaches –2 as x approaches a.

3. In question 1, you were asked to find the slopes of the graphs of **f** and **g.** What is the ratio of the slope of the graph of **f** to the slope of the graph of **g?** How does this compare to the limit you found in question 2?

Answer: The ratio of the slopes is also –2. It is the same as the limit.

4. Notice the zooming tool to the left of the screen shows that you are "zoomed in" as much as allowed by the tool. Use the slider to zoom out on the graph.



a. It turns out that these graphs were not straight lines. Why did they look like straight lines when you were zoomed in?

Answer: Answers may vary. Assuming the functions are differentiable, they would behave very much like linear functions "up close." Students may use the term "local linearity."

b. The zoomed-in version of the graph allowed us to approximate the slopes of the graphs near $x = a$. What is another name for the slope of a function at a point?

Answer: The derivative at that point.

c. Why does this suggest that $\lim_{x \to a} \frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ is the same as $\lim_{x \to a} \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)}$? Explain your reasoning.

Answer: In the zoomed-in window, we saw that the ratio of the y values was the same as the ratio of the slope values. Evidently, as we get very close to $x = a$, the ratio of the function values is very nearly the same as the ratio of the derivative values.

d. Based on your response to part 4b, what is $\lim_{x \to a} \frac{\mathbf{f}(x)}{\mathbf{g}(x)}$?

Answer: –2

5. What you have seen is the geometry behind something called l'Hôpital's Rule. L'Hôpital's Rule states, in part, that for differentiable functions **f** and **g,** with $\lim_{x\to a} \mathbf{f}(x) = 0$ and $\lim_{x\to a} \mathbf{g}(x) = 0$, then $\lim_{x\to a} \frac{\mathbf{f}(x)}{\mathbf{g}(x)} = \lim_{x\to a} \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)}$ if the limit of the quotients of the derivatives exists. Use l'Hôpital's Rule to find the following limits, if possible.

   a. $\lim_{x\to 1} \frac{\ln x}{x-1}$

   Answer: $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

   b. $\lim_{x\to -1} \frac{x^6-1}{x^4-1}$

   Answer: $\lim_{x\to -1} \frac{x^6-1}{x^4-1} = \lim_{x\to -1} \frac{6x^5}{4x^3} = \frac{-6}{-4} = \frac{3}{2}$

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- That limits of quotients of functions of the indeterminate form $\frac{0}{0}$ can be found by taking limits of quotients of the functions' derivatives, when those derivatives are not both 0.
- Local linearity of functions justifies the use of l'Hôpital's Rule.

Assessment

Have students find other limits requiring l'Hôpital's Rule, and provide opportunities to discuss when l'Hôpital's Rule cannot be applied. For example, use the actual functions **f** and **g** shown in the .tns file. **f**(x) = sin(x – 2) and **g**(x) = **f2**(x) = (x – 2)^2 – 0.5(x – 2)

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Note 1

Question 2a, *Quick Poll* (*Multiple Choice* or *Open Response*): Use a Quick Poll to find students' determinations of the limit. Opportunities may arise to discuss evaluation of limits for which limit laws cannot be used to simplify the function.

Note 2

Question 5b, *Quick Poll* (*Multiple Choice* or *Open Response*): Write a Quick Poll question for 5b. Multiple choice options could include 0, does not exist, 1, 3/2, 30.