Logos Arithmos

Teacher Notes and Answers

7 8 9 10 11 **12**

TI-Nspire™ Investigation Student 50 min

Introduction

The word *logos* is derived from ancient Greek and Latin and means 'proportion'; *arithmos* means 'number'. In 1614, mathematician John Napier published his book "Mirifici logarithmorum Canonis descriptio" which demonstrated how time consuming calculations involving multiplication could be changed to simple addition. An English translation of Napier's work was published two years later and the word Logarithm formed. The original intention may have been to simplify calculations which can now be preformed easily on a basic calculator, however, logarithms have many more uses and applications. This investigation focuses on understanding logarithms beyond a button on the calculator!

Investigation

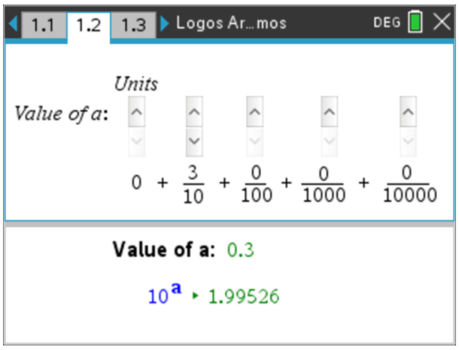
Suppose we want to find the value of x such that: $10^x = 100$. We know that $10^2 = 100$ so it follows that $x = 2$. Similarly if we are trying to solve: $10^x = 1000$ then it follows that $x = 3$ since $10^3 = 1000$. The problems can get much more complicated when the answer is not an integer. Suppose we want to solve: $10^x = 500$. It is reasonable to assume that the answer lies between 2 and 3. It may be tempting to think $x \approx 2.5$, but $10^{2.5} \approx 316$. The notion that x is approximately half way between 2 and 3 since 500 is approximately half way between 100 and 1000 is a linear way of thinking. We are working with exponentials. Exponentials are not linear.

The TI-nspire file: Logos Arithmos contains a series of interactive environments to help understand logarithms, starting with a series of questions similar to the above problem where trial and error can be used to determine the answers followed by a good working knowledge of index laws.

Open the TI-nspire document: Logos Arithmos.

Navigate to Page 1.2

The aim is to complete the table in Question 1 by adjusting the sliders one at a time until the target result is achieved.



Question: 1.

Determine the value of ***a*** for each of the following (below) where 10^a = target. Note that some values are not required (greyed out). The solutions for $10^a = 1$ and $10^a = 2$ are already completed. Check that these are correct before proceeding.

Target:	1	2	3	4	5	6	7	8	9	10
Value of ***a***	0	0.3010	0.4771		0.6990		0.8451			

Author: Peter Fox

Question: 2.

Given that 2 x 2 = 4 and 2 ≈ $10^{0.3010}$, it follows that $10^{0.3010}$ x $10^{0.3010}$ ≈ 4.

Use index laws to determine the value of ***a*** such that 10^a = 4. Check your answer using the setup on Page 1.2

Answer: Index law: $a^m \times a^n = a^{m+n}$ so it follows that $10^{0.3010} \times 10^{0.3010} = 10^{0.6020} \approx 4$ therefore $a \approx 0.6020$

Question: 3.

Given 2 x 3 = 6, use 2 ≈ $10^{0.3010}$ and your approximate value for 3 in the form 10^x to find the solution for 10^x = 6.

Answer:

$$2 \times 3 = 6$$
$$10^{0.3010} \times 10^{0.4771} \approx 6$$
$$10^{0.7781} \approx 6$$

Question: 4.

Given that 2^3 = 8 and 2 ≈ $10^{0.3010}$, determine an approximate solution for 10^x = 8, then complete the remainder of the table in Question 1.

Answer:

$$2^3 = 8$$
$$\left(10^{0.3010}\right)^3 \approx 8$$
$$10^{0.9030} \approx 8$$

Question: 5.

Suppose we want to calculate 45 x 14 without actually doing any multiplication.

a. Given: 45 = 9 x 5, express 45 in the form: 10^x.

Answer:

$$45 = 10^x$$
$$3^2 \times 5 = 10^x$$
$$\left(10^{0.4771}\right)^2 \times 10^{0.6990} \approx 10^x$$
$$45 \approx 10^{1.6532}$$

b. Given 14 = 2 x 7 express 14 in the form: 10^x.

Answer:

$$14 = 10^x$$
$$2 \times 7 = 10^x$$
$$10^{0.3010} \times 10^{0.8451} \approx 10^x$$
$$14 \approx 10^{1.1461}$$

c. Use the above results to express 45 x 14 in the form: 10^x.

Answer:

$$45 \times 14 = 10^x$$
$$3^2 \times 5 \times 2 \times 7 \approx 10^x$$
$$\left(10^{0.4771}\right)^2 \times 10^{0.6990} \times 10^{0.3010} \times 10^{0.8451} \approx 10^x$$
$$10^{0.9542+0.6990+0.3010+0.8451} \approx 10^{2.7993}$$

Author: Peter Fox

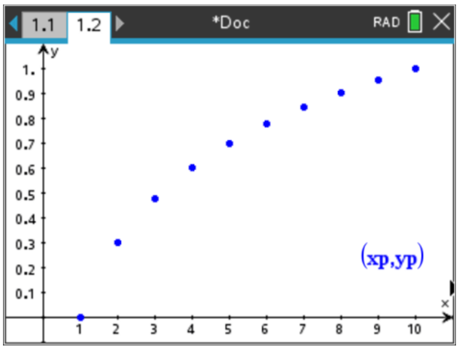
d. Compare the answer calculated using the above process and the actual answer. Explain any differences.

$10^{2.7993} \approx 629.94$ which is very close to the actual answer 45 x 14 = 630, the difference occurs due to rounding. A more accurate result: $10^{2.7993405494535}$ shows that the computed answer using index laws is correct to 4 decimal places. Just one additional decimal place: $10^{2.79934} \approx 629.999$

Question: 6.

Graph the values from the table in Question 1 and determine if the relationship is linear.

Answer:



Question: 7.

Complete the following table of values making use of values already determined and corresponding index laws wherever possible. Comment on which numbers could not be established using previous calculations.

Target:	11	12	13	14	15	16	17	18	19	20
Value of ***a***	1.04139	1.07918	1.11394	1.14613	1.17609	1.20412	1.23045	1.25527	1.27875	1.30103

Suppose we have a slightly different problem: $2^x = 5$. Is it possible to use our existing tables to determine a solution? The only difference is the base (2). We can use the index law:

$$\left(a^y\right)^x = a^{xy}$$

From the table in Question 1 we know that $10^{0.3010} \approx 2$ and $10^{0.6989} \approx 5$. This means that $(10^{0.3010})^x = 10^{0.6989}$.
Using the index law above we need to solve: $0.3010x = 0.6989$.

Question: 8.

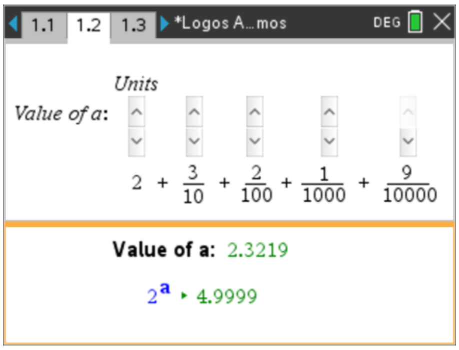
Determine the value of ***a*** for each of the following (below) where 2^a = target. Use the method outlined above, then check your answers using the calculator by changing the 10 on page 1.2 to a 2.

Target:	1	2	3	4	5	6	7	8	9	10
Value of ***a***	0	1	1.5850	2	2.3219	2.5850	2.8074	3	3.1699	3.3219

Author: Peter Fox

Note in the application shown opposite the target is now of the form: 2^a.

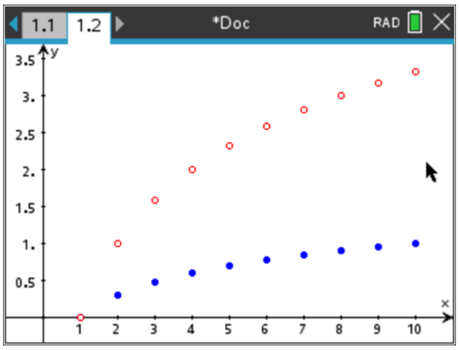
To edit this value click on the original base (10) and change the base value to: 2.



Question: 9.

Graph the table in Question 7 on the same set of axis as Question 6. Comment on the similarities and differences between the two sets of data.

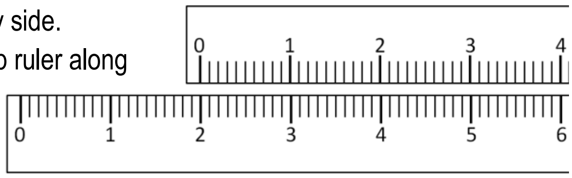
Answer:



The table of values generated in Questions 1, 7 & 8 are referred to a 'Log Tables'. Napier's original book included 50 pages of such tables. The graphs represent graphs of logarithmic functions with base 10 (Question 6) and base 2 (Question 9).

The Slide Rule

Not long after Napier created logarithms, the slide rule was invented. To help understand the slide rule, start with two simple rulers placed side by side. Start with both rulers aligned at zero, to calculate 2 + 3, slide the top ruler along so the zero lines up with 2 on the bottom ruler. Look along the top ruler until you find 3. The 3 is lined up with 5, so 2 + 3 = 5. This seems 'straight' forward.



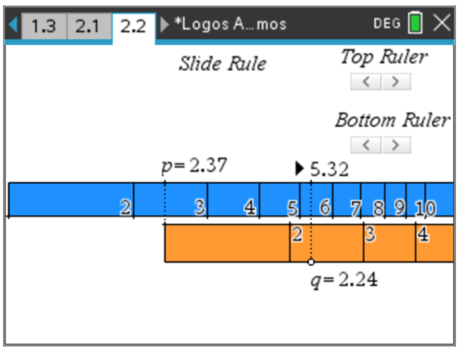
Calculations for exponents are not linear. The scales on a slide rule are non-linear, but the concept is essentially the same. A virtual slide rule (logarithmic scale) is located on Page 2.2.

The top ruler can be moved to the left/right using the "Top Ruler" slider. The movement is relatively significant.

The bottom ruler can be moved left/right also. The movements of the bottom ruler are much finer.

Use the sliders to locate the first number in the multiplication problem. In the example shown opposite this would be: 2.37.

Drag point q along to represent the second number then read the answer



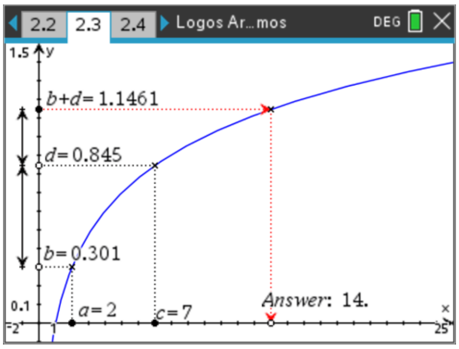
back off the top ruler.

Example: 2.37 x 2.24 ≈ 5.32 [Shown opposite]

Question: 10.

Use the virtual slide rule on Page 2.2 to calculate approximate answers to the following:

a) 3.2 x 2.6 = 8.32 b) 32 x 26 = 100 x 8.32 c) 4.2 x 1.9 = 7.97



A graphical version of the slide rule is located on Page 2.3. The graph shown is $y = \log_{10}(x)$ and is representative of the table of values in Question 1 where the base = 10.

Notice that the scale on the x axis is very different to the one on the y axis. Drag point ***a*** along the x axis. The values generated from the graph should be the same as the table generated in Question 1. That is: $\boldsymbol{b = \log_{10}(a)}$.

Point C can be moved such that $\boldsymbol{d = \log_{10}(c)}$.

b* + *d is automatically computed where: $\boldsymbol{b + d = \log_{10}(a) + \log_{10}(c)}$.

The result is returned to the x axis to determine the product of ***a*** and ***c***.

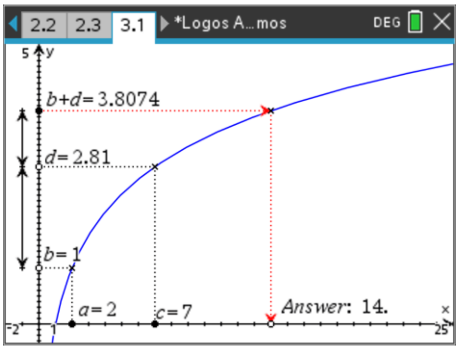
The final answer is looked up in the opposite or inverse direction.
In the example shown this means: $10^{1.1461} = 14$

Question: 11.

Use the graphical slide rule on page 2.3 to calculate the product of 3 and 6. (3 x 6).
Write your equation in the form: $\log_{10}(a) + \log_{10}(c) = \ldots$

$$\log_{10}(3) + \log_{10}(6) \approx 1.2553$$
$$10^{1.2553} \approx 18$$

Note that it also follows from the second line that: $\log_{10}(3) + \log_{10}(6) = \log_{10}(18)$



Navigate to Page 3.1. Another graphical version of the slide rule is located on Page 2.3. The graph shown is $y = \log_2(x)$ and is representative of the table of values in Question 8 where the base = 2.

Notice that the y axis scale is different than the previous graph and still very different from the x axis scale. Notice also that the values of ***b*** and ***d*** are also different but the overall result is still the same.

In this example: $\boldsymbol{b + d = \log_2(a) + \log_2(c)}$.

The reverse calculation therefore means: $2^{3.8074} \approx 14$.

Question: 12.

Use the graphical slide rule on Page 3.1 to calculate the product of 3 and 6. (3 x 6).
Write your equation in the form: $\log_2(a) + \log_2(c) = \ldots$ and compare with the calculations from Question 11.

Answer: $\log_2(3) + \log_2(6) \approx 1.585 + 2.58$
$$2^{4.165} \approx 18$$

The procedure is the same, as the base is different the numerical calculations will be different. Of course the overall result is still the same: 3 x 6 = 18

Author: Peter Fox

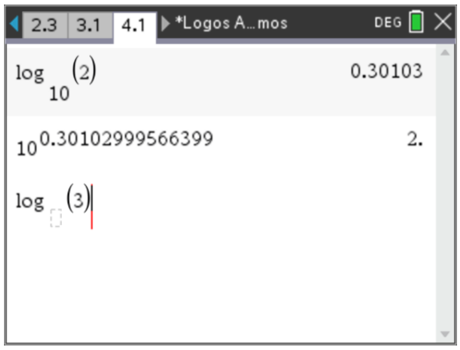
Putting it all together

Even though we have calculators that can multiply and divide numbers very quickly, logarithms are still used to solve problems such as: $10^x = 5$. Logarithms have numerous applications, based mostly on the same concept of turning multiplication into addition, linearising something that is non-linear.

Navigate to Page 4.1.

Logarithms of any base can be computed by the calculator by pressing:

Ctrl + 10^x The Ctrl key activates the second function: log(x)



Enter the base (skipping the base will default to log base 10) followed by the value for which the logarithm is to be computed. Start by checking some of the values computed in Table 1.

Question: 13.

Calculate each of the following:

a) $\log_5(25)$ = 2 b) $\log_3(9)$ = 2 c) $\log_6(36)$ = 2

Question: 14.

Use your answers to the previous questions to solve each of the following:

b) $\log_n(49) = 2$ n = 7 b) $\log_{13}(n) = 2$ n = 169 c) $\log_a(a^n) = 2$ n = 2

Question: 15.

Re-write question 14 as indicial expressions.

Answers: a) $5^2 = 25$ b) $3^2 = 9$ c) $6^2 = 36$

Teacher Notes:

The main purpose of this activity is to help students connect logarithms with indices; understand that just like indices, logarithms have different bases and that the laws associated with logarithms are naturally related to those associated with indices.

Author: Peter Fox