Elliptic Variations

Math Objectives

- Students will be able to describe the characteristics of the graphs of equations of the form $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$ for various values of n.
- Students will be able to describe the characteristics of the graphs of equations of the form $\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2} = b^2$ when a = 4 and b varies.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).
- Students will reason abstractly and quantitatively (CCSS Mathematical Practice).

Vocabulary

- ellipse
- superellipse
- Cassini oval

About the Lesson

- This lesson involves exploring the curves that result from varying the defining conditions of an ellipse.
- As a result, students will:
  - Analyze the properties of the graphs and equations of superellipses and Cassini ovals.
  - Analyze the derivations of the parametric representation of a superellipse and the polar representation of a Cassini oval.

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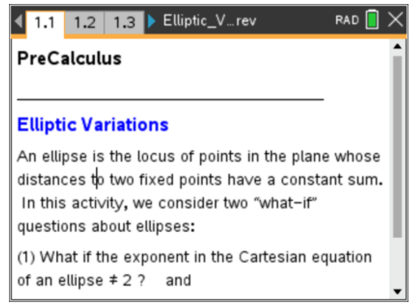
- Transfer a File.
- Use Screen Capture to monitor student progress.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
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- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:

Student Activity
Elliptic_Variations_Student.pdf
Elliptic_Variations_Student.doc

TI-Nspire document
Elliptic_Variations.tns

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Discussion Points and Possible Answers

Problem 1: Superellipses

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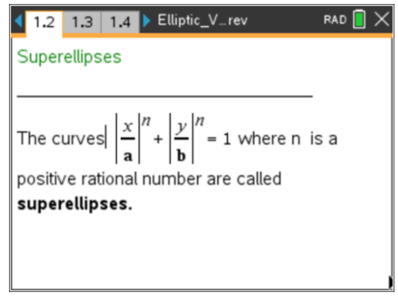

The curves described by the equation $\left|\frac{x}{a}\right|^n+\left|\frac{y}{b}\right|^n=1$, where n is a positive rational number, are called **superellipses**.

To conveniently explore fractional values of n, we set n = $\frac{k}{6}$ and consider the values of n from $\frac{1}{6}$ to 4, or equivalently, the values of k from 1 to 24.

Piet Hein popularized these curves around 1960 for values of n > 2 [especially n = 2.5]. The curves for these values of n are “rounded rectangles” that have been used in the design of a town “square” in Sweden to improve traffic flow and in the design of tabletops and other furniture.

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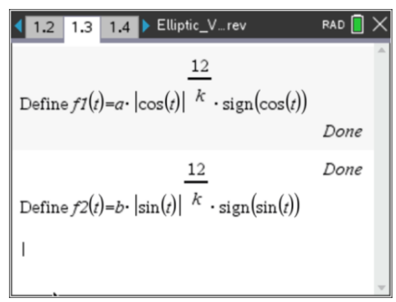

We will use parametric equations to produce a graph of these superellipses. The derivation of these equations is shown later in this section.

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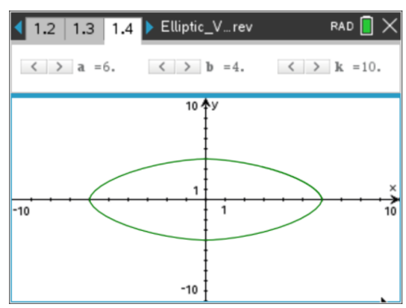

Use the a-clicker and b-clicker to set the values of a and b to a = 6 and b = 4 Using the k-clicker, scroll through the values of k = 1 to 24. As you answer the questions below, you might want to experiment with other values of a and b to confirm your responses.

1. Which value of k generates the graph of a standard ellipse?

   Answer: k = 12 since 12/6 = 2.

2. Which value of k generates the graph of a polygon, instead of a curve? Identify the type of polygon.

Answer: k = 6 since 6/6 = 1. The graph is a rhombus with sides of length $\sqrt{a^2+b^2}$ that are contained in the four lines whose equations are $\pm\frac{x}{a}\pm\frac{y}{b}=1$

3. Let k* denote the value of k in your answer to Question #2. Describe the graphs for 1 ≤ k ≤ k*.

Sample Answer. For 1 ≤ k ≤ 5, the graphs resemble 4-pointed stars whose sides curve inward.

4. Use the a-clicker and b-clicker to set the values of a and b to a = 6 and b = 6 Using the k-clicker, scroll through the values of k = 1 to 24. Describe the curve when $a = b$.

Answer: The sequence of graphs from k = 1 to k = 24 is similar to the set of graphs of the superellipses. The graph is a 4-pointed star whose sides curve inward for 1 ≤ k ≤ 5; a square for k = 6; an "oval" approaching a circle for 7 ≤ k ≤ 11; a circle for k = 12, and increasingly large "circular ovals" eventually approaching a square for k > 12. Such graphs are called **supercircles.**

5. What do you predict will happen to the graph of $\left|\frac{x}{a}\right|^n+\left|\frac{y}{b}\right|^n=1$ as n becomes very large? Why?

Sample Answer: As n gets very large in $\left|\frac{x}{a}\right|^n+\left|\frac{y}{b}\right|^n=1$ the four "corner points" of the graph approach, but do not reach, the points $(\pm a,\pm b)$ so the graph approaches a rectangle of area $2a\cdot 2b=4ab$.

6. What do you predict will happen to the graph of $\left|\frac{x}{a}\right|^n+\left|\frac{y}{b}\right|^n=1$ as n becomes very small? Why?

Sample Answer: As n approaches 0, either $\left|\frac{x}{a}\right|\to 0$ or $\left|\frac{y}{b}\right|\to 0$, but not both, since $|w|^0=1$ for any non-zero value of w. Thus the graph approaches the two coordinate axes without the origin – the set of points (x,0) and (0, y) omitting (0,0).

TI-Nspire Navigator Opportunity: ***Screen Capture*** **See Note 1 at the end of this lesson.**

Recall that the parametric equations for the standard ellipse are $x = a \cdot \cos(t)$ and $y = b \cdot \sin(t)$ for $0 \le t \le 2\pi$.

To modify these equations for a superellipse, we have $\begin{cases} x = \pm a \cdot |\cos(t)|^{2/n} \\ y = \pm b \cdot |\sin(t)|^{2/n} \end{cases}$.

These equations are valid because $\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = \left(|\cos(t)|^{2/n}\right)^n + \left(|\sin(t)|^{2/n}\right)^n$
$= (\cos(t))^2 + (\sin(t))^2 = 1$.

The difficulty with using these equations is that there are four of them. We would need to draw four different curves corresponding to the four possible combinations of + (plus) and - (minus). Fortunately, we can incorporate the **signum** function, $sign(x)$, into these equations to combine all of them into one equation. The signum function is defined by

$$sign(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

Note: On the calculator, $sign(0) \ne 0$. This anomaly does not affect the graphs in this problem.

The parametric equations for a superellipse become

$$\begin{cases} x = a \cdot |\cos(t)|^{2/n} \cdot sign(\cos(t)) \\ y = b \cdot |\sin(t)|^{2/n} \cdot sign(\sin(t)) \end{cases} \quad \text{for } 0 \le t \le 2\pi.$$

7. Explain carefully how these equations determine the set of points on the graph in the second quadrant when $\frac{\pi}{2} \le x \le \pi$.

   <u>Sample Answer</u>: For $\frac{\pi}{2} \le x \le \pi$, $sign(\cos(t)) = -1$ and $sign(\sin(t)) = 1$. Thus $x < 0$ and $y > 0$, so that (x, y) is in Quadrant II.

> **Teacher Tip:** (1) You can omit this derivation and Question #7 if time is short. (2) See the activity: Parametric Equations for Conic Sections.

Extension

- Design an item using one or more supercircles or superellipses.

Problem 2: Cassini Ovals

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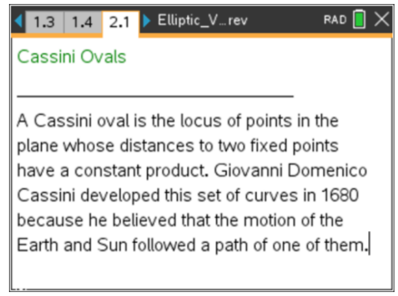

A **Cassini oval** is the locus of points in the plane whose distances to two fixed points have a constant product. Giovanni Domenico Cassini developed this set of curves in 1680 because he believed that the motion of the Earth and Sun followed the path of a Cassini oval.

In other words, a Cassini oval is the graph of the equation $\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=b^2$ for various values of a and b. The fixed points are ($-a$,0) and (a,0), and the distance is b^2. In this activity, we consider the case when a = 4.

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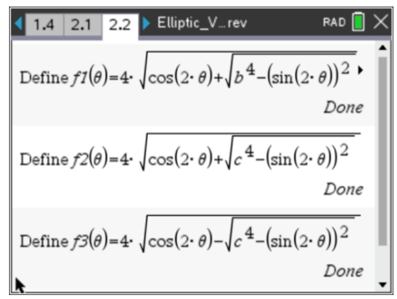

We will use polar equations to produce graphs of Cassini ovals. The derivation of these equations is shown later in this section.

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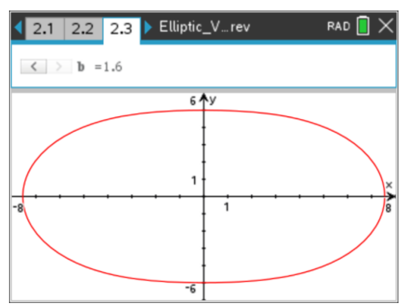

Set the value of the b-clicker to b = 1.6. Scroll through the values for b = 1.6 down to b = 1.0.

8. Describe how the curve changes for values of b between 1.6 and 1.0.

   <u>Sample Answer</u>: For b = 1.6 (and greater) the curve is “ellipse-like”. As b gets closer to 1, the curve becomes concave and resembles a “dog-bone”. When b = 1, the curve is a “figure-8” called a **lemniscate**.

9. What do you predict will happen to the curve as b becomes very large? Why?

   <u>Sample Answer</u>: The curve becomes an arbitrarily large oval. The product of distances can be arbitrarily large so the distance from a given point in the plane to each of the fixed points (-4,0) and (4,0) can be arbitrarily large.

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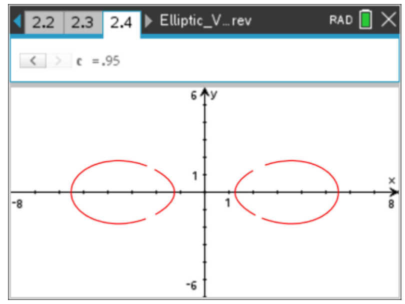

Set the value of the c-clicker to c = 0.95. Scroll through the values for c = 0.95 down to c = 0.5.

Note: Because of an anomaly in the way the calculator draws such graphs, the graphs of the ovals appear to be open, or disconnected, but, in fact, they are closed, connected graphs.

10. Describe how the curve changes for values of c between 0.95 and 0.5.

    <u>Sample Answer:</u> The curve consists of two loops that get smaller as the value of b decreases.

11. What do you predict will happen to the curve as c becomes very small?

    <u>Sample Answer:</u> The curve approaches the two points (-4,0) and (4,0). If the product of the distances from a point to the two fixed points is 0, $\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=0$, then one of those distances must be 0. Such a distance is 0 only when the ordered pair corresponds to one of the fixed points.

To derive the general polar equations of $\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=b^2$, we start by letting $x=r\cos\theta$ and $y=r\sin\theta$. Then

(1) $[(r\cos\theta-a)^2+(r\sin\theta-a)^2]\cdot[(r\cos\theta+a)^2+(r\sin\theta+a)^2]=b^4$

(2) $r^4[\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta]-2a^2r^2[\cos^2\theta-\sin^2\theta]+a^4=b^4=$

(3) $r^4-2a^2r^2\cos 2\theta+a^4=b^4$

Solving for r^2 and then r gives

(4) $r^2=a^2\cos 2\theta\pm\sqrt{a^4\cos^2 2\theta-a^4+b^4}$ or

(5) $r^2=a^2\left(\cos 2\theta\pm\sqrt{\left(\frac{b}{a}\right)^4-\sin^2 2\theta}\right)$ so that

(6) $r=\pm a\sqrt{\cos 2\theta\pm\sqrt{\left(\frac{b}{a}\right)^4-\sin^2 2\theta}}$.

12. Explain how to go from step (2) to step (3) and from step (4) to step (5) in the argument above.

Sample Answer: Use trig identities:
step (2) to step (3) : $[\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta]=[\cos^2\theta+\sin^2\theta]^2=1^2=1$;
$[\cos^2\theta-\sin^2\theta]=\cos 2\theta$;

step (4) to step(5): $\sqrt{a^4\cos^2 2\theta-a^4+b^4}=\sqrt{b^4-a^4(1-\cos^2 2\theta)}=a^2\sqrt{\left(\frac{b}{a}\right)^4-\sin^2 2\theta}$

Teacher Tip: (1) You can omit this derivation and Question #12 if time is short. (2) See the activity: Polar Conics.

Extension

- Research lemniscates and other well-known polar curves.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- How to analyze properties of superellipses and Cassini ovals.
- How to think about the extreme cases of a situation, e.g. when the values of the parameters are very large or very small.

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Note 1

Name of Feature: Screen Capture
Use Screen Capture to monitor and discuss the choices that students make for the values of $a, b, and\ k$ and how they reach their answers – especially to Questions 5 and 6.