Monday Night Calculus

Slope Fields and Differential Equations

Exercises

1. The indefinite integral (antiderivative) formula

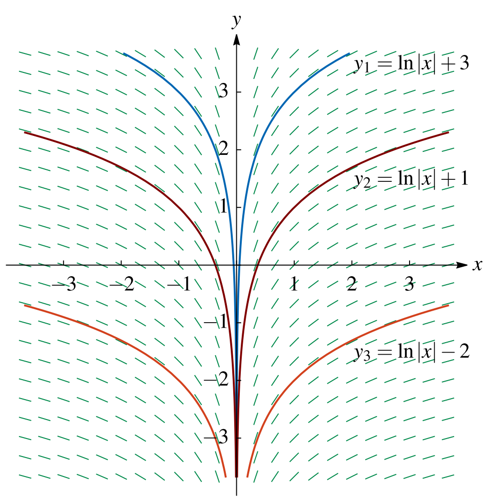
$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{where } C \text{ is an arbitrary constant}$$

is found in the inside cover of almost every calculus book. We can interpret this result as the general solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$

which led mathematician David Tall to write an article called "Lies, Damned Lies, and Differential Equations."

(a) Sketch a slope field for the differential equation $\frac{dy}{dx} = \frac{1}{x}$ and the three functions $y_1 = \ln|x| + 3$, $y_2 = \ln|x| + 1$, and $y_3 = \ln|x| - 2$ on the same coordinate axes. Are these three functions solutions to the differential equation for all nonzero x values?



The three functions are solutions to the differential equation for all nonzero x values.

(b) Find a solution to the differential equation that is valid for all nonzero x values, but is not of the form $y = \ln|x| + C$.

Here is a solution: $f(x) = \begin{cases} \ln(-x) + 1 & \text{if } x < 0 \\ \ln(x) - 2 & \text{if } x > 0 \end{cases}$

Note that we can choose different constants for the two parts of the domain: $x < 0$ and $x > 0$.

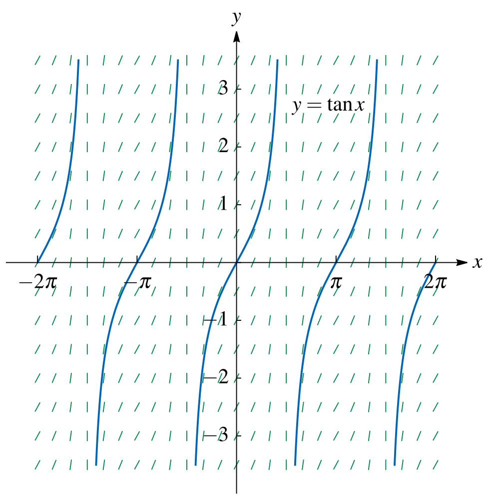
2. **(a)** For a differential equation of the form $\frac{dy}{dx} = f(x)$, the line segments in the slope field in any vertical column will all have the same slope. Similarly, for a differential equation of the form $\frac{dy}{dx} = g(y)$, the line segments in the slope filed in and horizontal row will all have the same slope. Explain why.

If $\frac{dy}{dx} = f(x)$, all the segments in a vertical column are centered at the same x-value and, therefore, will have the same slope $f(x)$ for that x-value.

Similarly, if $\frac{dy}{dx} = g(y)$, all the segments in a horizontal row are centered at the same y-value and, therefore, will have the same slope $g(y)$ for that y-value.

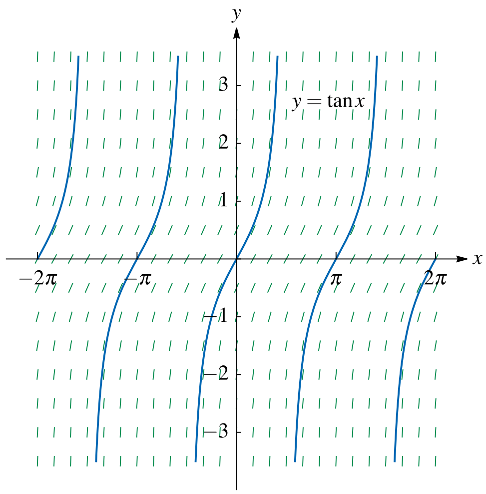
(b) Sketch a slope field for the differential equation $\frac{dy}{dx} = \sec^2 x$.

Find a solution $y = f(x)$ to this differential equation whose graph passes through the origin.



(c) Sketch a slope field for the differential equation $\dfrac{dy}{dx} = 1 + y^2$.

Show that the function found in part (b) is also a solution to this differential equation whose graph passes through the origin.



(d) Find the general solution to the differential equation $\dfrac{dy}{dx} = \sec^2 x$.

$$dy = \sec^2 x \, dx$$

$$\int dy = \int \sec^2 x \, dx$$

$$y = \tan x + C$$

(e) Use separation of variables to find the general solution to the differential equation $\dfrac{dy}{dx} = 1 + y^2$. Compare this solution with the general solution found in part (d).

$$\frac{dy}{1 + y^2} = dx$$

$$\int \frac{dy}{1 + y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$y = \tan(x + C)$$

Note the difference in the role of the arbitrary constant C.

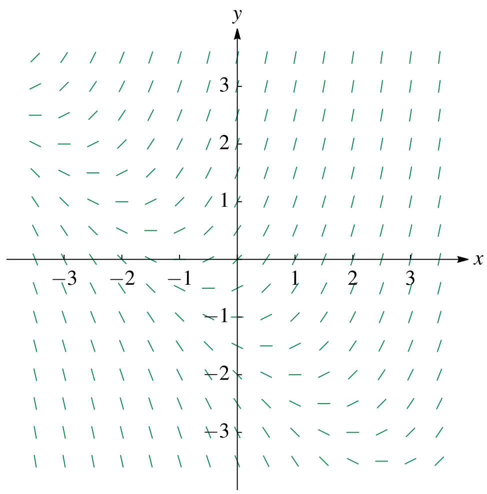
One results in a vertical shift and the other a horizontal shift!

3. Match each differential equation (A)-(F) with a slope field (I)-(VI).

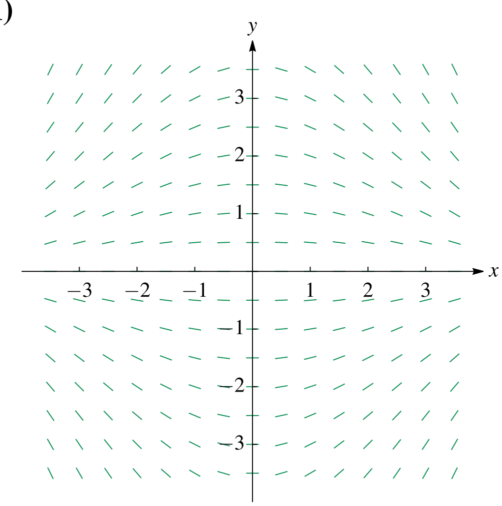
(A) $\dfrac{dy}{dx} = e^{-x^2}$ **(B)** $\dfrac{dy}{dx} = \dfrac{y}{1+x^2}$ **(C)** $\dfrac{dy}{dx} = x + y + 1$

(D) $\dfrac{dy}{dx} = \dfrac{1}{y}$ **(E)** $\dfrac{dy}{dx} = \dfrac{-xy}{6}$ **(F)** $\dfrac{dy}{dx} = \dfrac{y(4-y)}{2}$

(I)



(II)



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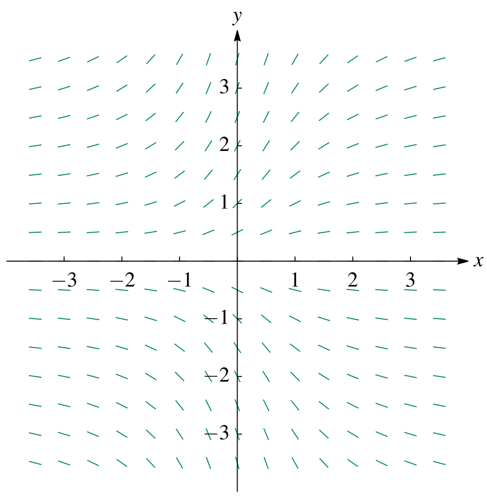

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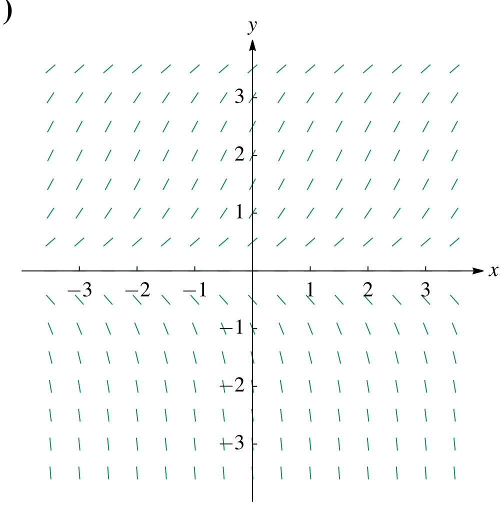
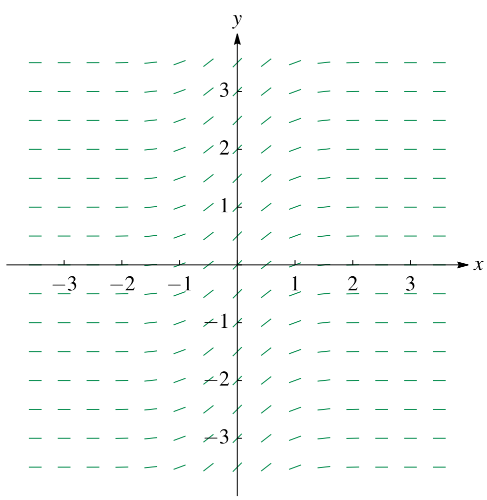
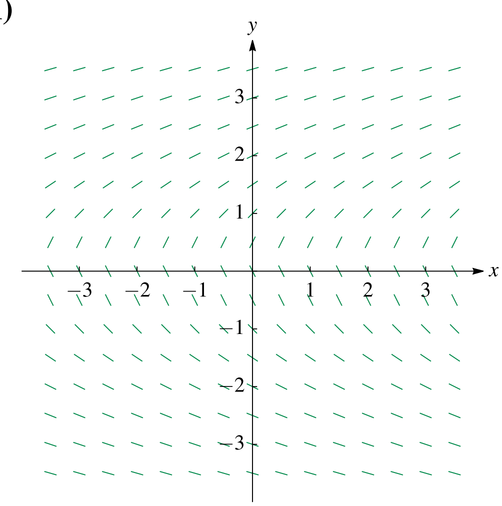


(V)



(VI)



(A) → (V) ; (B) → (III) ; (C) → (I) ; (D) → (VI) ; (E) → (II) ; (F) → (IV)